Problems And Solutions In Probability And

Problems and Solutions in

PROBABILITY AND STATISTICS

Dr. Smrati Rai
Professor and Head
Department of Humanities & Sciences
Teegala Krishna Reddy Engineering College
Hyderabad (A.P)

Publishing for future

SCITECH PUBLICATIONS (INDIA) PVT. LTD.

CHENNAI • HYDERABAD • KOLKATA • PUNE • BHUBANESWAR • NEW DELHI

SCITECH PUBLICATIONS (INDIA) PVT. LTD.

Corp. Office: No.4, Anna Street, Chitlapakkam, CHENNAI – 600 064
E-mail: scitechcorp@yahoo.co.in

Reg. Office: 7/3c, Madley Road, T.Nagar, CHENNAI-600 017
E-mail: scitech@md5.vsnl.net.in
Visit us at: www.scitechbooks.in

Branches:

BHUBANESWAR: scitech_bbsr@yahoo.co.in

CHENNAI: scitech@md5.vsnl.net.in

HYDERABAD: scitech_hyd@yahoo.co.in

KOLKATA: scitech_kol@yahoo.co.in

PUNE: scitech_pune@yahoo.co.in

NEW DELHI: scitech_del@yahoo.co.in

978 81 8371 447 1

Production Co-ordinator: K. Madhu Sudhan

Published by: M. R. Purushothaman for Scitech Publications (India) Pvt. Ltd.
3-4-249/1, Beside Venkataramana Theatre Lane, Street No.2, Lingampally,
Kachiguda, Hyderabad -500 027. Printed at: Shiva Sai Printers, Hyderabad.

I feel gi
Solutions in Pr
students. Proba
disciplines. Thi
& Engineering
in it is easy to u
covered are : P
Distribution, Sa
Queuing Theor
statements of d
descriptive and
solved examples
from past unive
to make the stud

I wish to
operation in brin

I shall b

PREFACE

I feel great pleasure in bringing out the present book "Problems & Solutions in Probability & Statistics", which meets the requirements of the students. Probability has applications in engineering as well as other disciplines. This book is designed, for graduate, post graduate, MCA, MBA & Engineering courses of various Indian Universities. The matter presented in it is easy to understand. This book consists of Eight chapters. The topics covered are : Probability, Random variables, Binomial, Poisson and Normal Distribution, Sampling, Test of Hypothesis, Correlation and Regression, Queuing Theory and Stochastic Process. Each chapter begins with clear statements of definitions, principles and theorems with proofs and other descriptive and objective material. Each chapter contains large number of solved examples, hence it can easily be used for self study. Model questions from past university examinations (JNTU) have been included in examples to make the students familiar with that type of questions.

I wish to thank Scitech Publications, for their keen interest and Co-operation in bringing out this book.

I shall be grateful for any suggestions for improvement of the book.

Smrati Rai

CONTENTS

"I hear, and

1.1 SAMPLE SP

A set of all p
denoted by S.

Example

1. In an experim
   Sample space
2. In an experime
   Sample space

1.2 EVENT

Any subset of a sar

Example

1. In an experime
an event.

$E_1 = \{H\}$ or E_2

UNIT-1

PROBABILITY

"I hear, and I forget. I see and I remember. I do, and I understand." - chinese proverb

1.1 SAMPLE SPACE

A set of all possible out comes of an experiment is called a sample space and denoted by S.

Example

1. In an experiment of tossing of a coin, possible out comes are Head or Tail.
   Sample space S = {H, T}
2. In an experiment of throwing a die, possible out comes are 1, 2, 3, 4, 5, or 6.
   Sample space S = {1, 2, 3, 4, 5, 6}

1.2 EVENT

Any subset of a sample space is called an event and denoted by E

Example

1. In an experiment of tossing of a coin, getting Head is an event or getting Tail is an event.

   $E_1 = \{H\}$ or $E_2 = \{T\}$

2. In an experiment of throwing a die, getting 1, 2, 3, 4, 5, or 6 are events.

$E_1 = \{1\}$, $E_2 = \{2\}$, $E_3 = \{3\}$, $E_4 = \{4\}$, $E_5 = \{5\}$, $E_6 = \{6\}$ getting 1 or 2 is also an event $E = \{1 \text{ or } 2\}$

1.3 MUTUALLY EXCLUSIVE EVENTS

Two events A and B are mutually exclusive if $A \cap B = \phi$ i.e. A and B are disjoint or A and B can not happen simultaneously.

Example

1. In throwing of a die all 6 possible cases are mutually exclusive.

1.4 EQUALLY LIKELY EVENTS

In the experiment of tossing of a coin the chance are half and half of getting Head or Tail. Both the events are equally likely to happen.

1.5 COLLECTIVELY EXHAUSTIVE EVENTS

If the events are A_1, A_2 .... A_n are said to be collectively exhaustive if

$$\bigcup_{i=1}^{n} A_i = S$$

1.6 PERMUTATION

A permutation of a number of objects or collection is the arrangement of objects in some definite order.

Example

A = {a, b, c}

Then the possible arrangements are abc, bac, acb, bca, cab, and cba.

We see that any of the 3 letters can come on the first place, by 3 ways. After filling the first place, the second place can be filled in 2 ways. (by any of the remaining 2 letters) remaining one letter will come on third place by one way.

Hence the total number of arrangements of 3 objects are $3 \times 2 \times 1 = 6$

In general, n distinct objects can be arranged in

$$n(n-1)(n-2)\ldots\ldots\ldots\ldots 3,2,1 \text{ Ways}$$

or

$$\angle n = n(n-1)(n-2)\ldots\ldots\ldots\ldots 3,\ 2,\ 1$$

The number of permutations of n distinct objects (or collection) by taken r at a time is $^{n}P_{r} = \dfrac{\angle n}{\angle n-r} = \dfrac{n!}{(n-r)!}$

1.7 COMBINATI

The number c

collection taken r a

Example

Three cards a

that they are all ca

Solution

3 cards can be dra

=

=

=

3 cards can be dra

Hence the required

1.8 PROBABILI

In a random e

equally likely case

then the probabilit

defined by

The number of out

1.7 COMBINATION

The number of combination is an unordered selection of n distinct objects are collection taken r at a time is $C(n,r) = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Example

Three cards are drawn at random from a pack of 52 cards. Find the probability that they are all cards of hearts.

Solution

3 cards can be drawn from a pack of 52 cards in ${}^{52}C_3$ ways

$$= \frac{52!}{3!(52-3)!} = \frac{52!}{3!\ 49!}$$

$$= \frac{52 \times 51 \times 50 \times 49!}{3!\ \ 49!}$$

$$= \frac{52 \times 51 \times 50}{3 \times 2 \times 1}$$

3 cards can be drawn from the 13 cards of hearts in ${}^{13}C_3$ ways

$$= \frac{13!}{3!(13-3)!} = \frac{13 \times 12 \times 11 \times 10!}{(3 \times 2 \times 1) \times 10!}$$

Hence the required probability $= \dfrac{{}^{13}C_3}{{}^{52}C_3} = \dfrac{13 \times 12 \times 11}{52 \times 51 \times 50}$

$$= \frac{1716}{132600} = 0.0129$$

1.8 PROBABILITY – THE AXIOMS OF PROBABILITY

In a random experiment, n is the number of exhaustive, mutually exclusive and equally likely cases and m of them are favorable to the happening of an event E then the probability of occurrence of E or probability of E denoted by P(E) is defined by

$$P(E) = \frac{m}{n}$$

The number of out comes which are not favorable to this event is n – m,

The probability of not happening the event E, denoted by

$$P(\overline{E}) = P(E^C) = \frac{n-m}{n}$$

$$P(E^C) = 1 - \frac{m}{n}$$

$$P(E^C) = 1 - P(E)$$

$$P(E) + P(E^C) = 1$$

Notation

1. P (A) denotes for the probability of happening of the event A.
2. $P(\overline{A})$ or $P(A^C)$ or P(A not) denotes, the probability of not happening of the event A.
3. P (A+B) or P (A∪B) denotes, the probability of happening of the at least one of the events A and B (either A or B).
4. P (A.B) or P (A∩B) denotes, the probability of happening of both the events A and B.

1.9 AXIOMS OF PROBABILITY

1. For any event E of S

   $0 \le P(E) \le 1$ i.e. probability is a numerical value lying between 0 and 1.
2. P(S) = 1
3. If E_1 and E_2 are two mutually exclusive events in S, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

When the events E_1 and E_2 are not mutually exclusive then some favorable cases are there which favors both E_1 and E_2. Suppose this number is m_3. The number which favors to the event E_1 is m_1 and the number which favors to the event E_2 is m_2 .Hence the total number of outcomes which favors either event E_1or E_2 or both is

$m_1 + m_2 - m_3$.

Then the probability of happening the events E_1 and E_2 or both

$$P(E_1 + E_2) = P(E_1 \quad E_2) = \frac{m_1 + m_2 - m_3}{h}$$

$$= \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

$$= P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$P(E_1 + E_2)$ O
happening
$P(E_1 +$

$P(E_1 \cup$
$P(E_1.E_2)$ or
happening together.

For any sequen
$P(E_1 + E_2$

$P(E_1 \cup E_2 \cup$

1.10 SOME USEFU
Additive Theorem

1.10.1 Theorem: If
$P(A \cup$

1.10.2 Theorem
For any three events
is given by
P (A ∪B∪C) = P (A
P (

Proof: See Exampl
1.10.3 Theorem:

1.10.4 Theorem: If
$P\left(\bigcap_{i=1}^{n}\right.$

$P(E_1 + E_2)$ Or $P(E_1 \cup E_2)$ denotes the probability of either A or B or both happening

$$P(E_1 + E_2) = P(E_1) + P(E_2) - P(E_1.E_2)$$

OR

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$P(E_1.E_2)$ *or* $P(E_1 \cap E_2)$ denotes the probability of both E_1 and E_2 happening together.

For any sequence of mutually exclusive events

$$P(E_1 + E_2 + \dots\dots\dots + E_n) = P(E_1) + P(E_2) + \dots\dots\dots\dots + P(E_n)$$

OR

$$P(E_1 \cup E_2 \cup E_3 + \dots\dots\dots\dots + E_n) = P(E_1) + P(E_2) + P(E_3) + \dots\dots\dots P(E_n)$$

1.10 SOME USEFUL THEOREMS

Additive Theorem or Rule or General Addition Rule of Probabilities

1.10.1 Theorem: If A and B are any two arbitrary events of S then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

1.10.2 Theorem

For any three events A, B and C, the probability that at least one of them will occur is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

(Nov.2009 set 4)

(Supple Feb.2010 set 1)

Proof: See Example 1.49.on Page No.1.54

1.10.3 Theorem: If $B<A$ then $P(B) \leq P(A)$ **(Nov.2009 set 1**

1.10.4 Theorem: If $A_1, A_2 \dots\dots\dots A_n$ are n events then prove that

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

(Supple. Nov. /Dec. 2005)

Proof :

We know that for any two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If the events are A_1 and A_2 then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

According to the first axiom of probability, the probability of any event is greater than or equal to zero and less than or equal to 1. i.e.

$$P(A_1 \cup A_2) \leq 1$$

$$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$$

$$P(A_1) + P(A_2) \leq 1 + P(A_1 \cap A_2)$$

Or

$$1 + P(A_1 \cap A_2) \geq P(A_1) + P(A_2)$$

$$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(A)$$

For $A_1, A_2, A_3 \ldots A_r$ events

$$P(A_1 \cap A_2 \cap A_3 \ldots\ldots\ldots A_r) \geq P(A_1) + P(A_2) + \ldots\ldots P(A_r) - (r-1)$$

$$P\left(\bigcap_{i=1}^{r+1} A_i\right) \geq P\left(\bigcap_{i=1}^{r} A_i\right) + P(A_{r+1}) - 1 \text{ From the equation.......(A)}$$

$$\geq \sum_{i=1}^{r} P(A_i) - (r-1) + P(A_{r+1}) - 1$$

$$= \sum_{i=1}^{r+1} P(A_i) - r$$

$$\Rightarrow P\left(\bigcap_{i=1}^{r+1} A_1\right) \geq \sum_{i=1}^{r} P(A_i) - r$$

The hypothesis is true for n = r + 1. By the induction method

$$\boxed{P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)}$$

1.11 CONDITION

Conditional pr

defined as

The probability P(A

General multiplicati

Similarly

Conditional pro

defined as

Where P(B/A) is the

General multipl

1.12 INDEPENDE

Two events are

happening of the ot

Otherwise the events

Two events A and B

1.13 THEOREM O

If $B_1, B_2 \ldots\ldots B_n$

for any event A of S

$P(A) =$

1.14 STATE AND P

1.11 CONDITIONAL PROBABILITY

Conditional probability of an event A if B has happened, denoted by $P\left(\frac{A}{B}\right)$ is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \text{ If P (B)} \neq 0$$

The probability P(A/B) is known as conditional probability

General multiplication rule $P(A \cap B) = P(B).P(A/B) \quad P(B) \neq 0$

Similarly

Conditional probability of an event B if A has happened, denoted by P (B/A) is defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{if} \quad P(A) \neq 0$$

Where P(B/A) is the conditional probability.

General multiplication rule $P(A \cap B) = P(A).P(B/A) \qquad P(A) \neq 0$

1.12 INDEPENDENT EVENTS

Two events are said to be independent if happening of one does not affect the happening of the other or failure of one does not affect the failure of the other. Otherwise the events are said to be dependent. i.e.

$$P(B/A) = P(B)$$

$$P(A/B) = P(A)$$

Two events A and B are independent events if and only if

$$P(A \cap B) = P(A).P(B)$$

1.13 THEOREM ON TOTAL PROBABILITY

If B_1, B_2 .....B_n are mutually exclusive events of which one must happen then for any event A of S

$$P(A) = \sum_{i=1}^{n} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i).P(A/B_i)$$

1.14 STATE AND PROVE BAYE'S THEOREM.

(Supple.Nov. 2008)

(R07, Reg.Nov.2009, Set 2)

Statement If B_1, B_2 ....B_n are mutually exclusive events of which one must happen then

$$P(B_r/A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{n} P(B_i \cap A)}$$

$$P(B_r/A) = \frac{P(B_r).P(A/B_r)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$$

For r = 1, 2 ....n

1.15 RANDOM VARIABLE

If S is the sample space R is the set of real numbers then a random variable is a function from S to R ($X : S \to R$) that associates with each element in the sample space to a real number.

Example of random variable:

Two pens are drawn in succession without replacement from a box containing 4 red pens and 2 black pens. The possible out comes and the value y of the random variable Y, where Y is the number of red pens, is

Given that two pens are drawn in succession without replacement,

In a box there are 4 red pens and 2 black pens

1. Both two pens may be red
2. First pen may be red and second may be black
3. First black and second red
4. Both two pens may be black

	Random Variable:
First red, second red	2
First red, second black	1
First black, second red	1
First black, second black	0

1.15.1 Discrete Random Variable

A random variable is said to be discrete random variable if the sample space S (its set of possible out comes) is countable.

1.15.2 Continuous Random Variable

A random variable is said to be continuous random variable if the sample space S (its set of possible out comes) contains infinite numbers.

1.16 PROBABILI

1.16.1 Discrete Pr

The set of o
probability functio

It is also calle
the function f(x) sa

1. $f(x) \geq 0$
2. $\sum_{x} f(x) = 1$
3. P(X = x) = f(x

Example 1: Two p
black pens. Find th

Solution

Total number

1. Both two pens

2. One is black a

3. Both two pens

This probability dist

Example 2: In the
random variable the

Random variable X

Sample Space = {HH

1.16 PROBABILITY DISTRIBUTION

1.16.1 Discrete Probability Distribution

The set of ordered pairs (x, f(x)) is called the probability distribution or probability function of the discrete random variable X.

It is also called probability mass function, of a discrete random variable X is the function f(x) satisfying the following conditions.

1. $f(x) \geq 0$
2. $\sum_{x} f(x) = 1$
3. $P(X = x) = f(x)$

Example 1: Two pens are drawn at random from a box containing 4 red pens and 2 black pens. Find the probability distribution for the number of red pens.

Solution

Total number of pens is 6

1. Both two pens are black means number of red pens are zero

$$f(0) = P(X = 0) = \frac{2_{C_2}\, 4_{C_0}}{6_{C_2}} = \frac{1}{15} = \frac{1}{15}$$

2. One is black and one is red

$$f(1) = P(X = 1) = \frac{2_{C_1}\, 4_{C_1}}{6_{C_1}} = \frac{2 \times 4}{15} = \frac{8}{15}$$

3. Both two pens are red

$$f(2) = P(X = 2) = \frac{2_{C_0}\, 4_{C_2}}{6_{C_2}} = \frac{6}{15} = \frac{6}{15}$$

This probability distribution of X is

X	0	1	2
f(x)	$\frac{1}{15}$	$\frac{8}{15}$	$\frac{6}{15}$

Example 2: In the experiment of tossing of the three coins number of heads is the random variable then the probability distribution is

Random variable X = number of heads

Sample Space = {HHH, HTT, HTH, THH, THT, HHT, TTH, TTT}

Number of heads may be 0, 1, 2, or 3

Total number of out comes are eight

$X = x_i$	0	1	2	3
$f(x) = P(X = x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

1.16.2 Cumulative distribution

If the probability distribution is f(x) then the cumulative distribution F(x) of a discrete random variable X is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \quad for \ -\infty < x < \infty$$

Example of cumulative distribution function:

Suppose F(x) is the cumulative distribution function of a discrete random variable X is as fallows

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.3 & -2 \leq x < 0 \\ 0.8 & 0 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

Determine the discrete probability distribution function f(x) of the discrete random variable X.

Solution

We know that from the definition of the cumulative distribution function F(X), discrete random variable X are -2, 0 and 2 and by the definition

f(-2) + f(0) + f(2) = 1 ....(A)

f (-2) + f(0) = 0.8

From (A)   0.8 + f(2) = 1

f (2) = 1-0.8

$\boxed{f(2) = 0.2}$

$\boxed{f(-2) = 0.3}$

From (A) 0.3 + f (0) + 0.2 = 1

f(0) = 1 – 0.5 = 0.5

$x = x_i$	-2	0	2
f(x)	0.3	0.5	0.2

Example of cumula

The discrete probabi
coins is as given belc
Find the cumulative

f(x) =

1. If $-\infty < x < 0$ the
2. If $0 \leq x < 1$ then
3. If $1 \leq x < 2$ then
4. If $2 \leq x < 3$ then F
5. If $3 \leq x < \infty$ then

1.16.3 Continuous Pr

For the continuo
function or density fun

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a < x < b) = \int_a^b$

The cumulative d
density function f(x) the

Example of cumulative distribution function:

The discrete probability distribution, in the experiment of tossing of the three fair coins is as given below.

Find the cumulative distribution function of the random variable X.

X = x	0	1	2	3
f(x) = P(X = x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

1. If $-\infty < x < 0$ then $F(x) = 0$
2. If $0 \le x < 1$ then $F(x) = \frac{1}{8}$
3. If $1 \le x < 2$ then $F(x) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$
4. If $2 \le x < 3$ then $F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$
5. If $3 \le x < \infty$ then $F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{8} & 0 \le x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{7}{8} & 2 \le x < 3 \\ 1 & 3 \le x < \infty \end{cases}$$

1.16.3 Continuous Probability Distribution

For the continuous random variable, f(x) is called the probability density function or density function defined over the set of real number R, if

1. $f(x) \ge 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a < x < b) = \int_a^b f(x)dx$

The cumulative distribution, if X is the continuous random variable with density function f(x) then the cumulative distribution F(x) is

$$F(X)=P(X\le x)=\int_{-\infty}^{x} f(t)\,dt \quad for\ -\infty<x<\infty$$

1.17 MEAN OR EXPECTED VALUE

If X be a random variable and probability distribution is f(x) then expectation or mean or expected value of a random variable X denoted by E(X) or μ is defined as

$$\mu = E(X) = \sum_{x} x\, f(x) \quad \text{If X is discrete}$$

And

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx \quad \text{If X is continuous}$$

1.18 VARIANCE

$$\sigma^2 = E\left[(X-\mu)^2\right] = \sum_{x} (x-\mu)^2 f(x) \quad \text{If X is discrete}$$

Or

$$\sigma^2 = E(X^2)-\mu^2$$

$$\sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{If X is continuous}$$

1.19 STANDARD DEVIATION

If σ^2 is the variance then positive square root of variance σ^2 i.e. σ is called the standard deviation.

1.20 DISCRETE UNIFORM DISTRIBUTION

If the random variable assumes each of its value with an equal probability then probability distribution is called a discrete uniform distribution. Suppose the random variable X assumes the values $x_1, x_2, \ldots\ldots\ldots x_k$ with same (uniform) probability then discrete uniform distribution is

$$f(x)=\frac{1}{k}$$

x	x_1	x_2………………… x_k
f(x)	$\frac{1}{k}$	$\frac{1}{k}$ …. ………….. $\frac{1}{k}$

Mean of the discrete uniform distribution

$$\mu = E(X) = \sum_{x} x\, f(x)$$

$$= \sum_{i=1}^{k} x_i \, f(x_i)$$

$$= \sum_{i=1}^{k} x_i \frac{1}{k}$$

$$= \frac{\sum_{i=1}^{k} x_i}{k}$$

Variance of the discrete uniform distribution

$$\sigma^2 = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 \; f(x)$$

$$= \sum_{i=1}^{k} (x_i - \mu)^2 \, f(x_i)$$

$$= \sum_{i=1}^{k} \frac{(x_i - \mu)^2}{k}$$

1.21 CONTINUOUS UNIFORM DISTRIBUTION

If X is the continuous random variable and the probability density function f(x), it is uniformly distributed in the interval [a, b] is given by

$$f(x) = \begin{cases} k & a \le x \le b \\ 0 & other\ wise \end{cases}$$

We have $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_a^b f(x)\, dx = 1$$

$$\int_a^b k \; dx = 1$$

k (b – a) = 1 $\qquad$ k = $\dfrac{1}{b-a}$

Now the continuous uniformly distribution

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & other\ wise \end{cases}$$

Now for any subinterval [c, d] $\qquad$ Where $a \le c < d \le b$

The probability that x lies in the interval [c, d] is given by

$$P(c \le x \le d) = \int_c^d f(x)\,dx$$

We have

$$P(a \le x < b) = \int_a^b f(x)\,dx$$

$$P(c \le x \le d) = \int_c^d \frac{1}{b-a}\,dx$$

$$P(c \le x \le d) = \frac{d-c}{b-a}$$

1.22 CHEBYSHEV'S THEOREM

Let μ be the mean and σ be the standard deviation of any random variable X, then the probability that R.V. X will assume a value within k the standard deviation of the mean is at least $1-\frac{1}{k^2}$.

$$P(\mu - k\sigma < X < \mu + k\sigma) = P(|X - \mu| < k\sigma) \ge 1 - \frac{1}{k^2}$$

1.22.1 Example: A random variable X has a mean 8 and variance 9 with an unknown probability distribution, find

$$P(+2 < X < 14) = ?$$

Solution

By the Chebyshev's Theorem

$$P(\mu - k\sigma < X < \mu + k\sigma) \ge 1 - \frac{1}{k^2}$$

Given that mean $\mu = 8$

Variance $\sigma^2 = 9$ $\qquad \sigma = 3$

$\mu - k\sigma = 2$	$\mu + k\mu = 14$
$8 - k(3) = 2$	$8 + k(3) = 14$
$-3k = 2 - 8$	$3k = 14 - 8$
$-3k = -6$	$3k = 6$
$k = 2$	$k = 2$

$$P(2 < X < 14) = P(8 - (2)(3) < X < 8 + (2)3)$$

$$\ge 1 - \frac{1}{k^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

P(2 < X

1.22.2 Example:
probability distrit
Solution
P(

μ - kσ = 6
12 – k (3) =
- 3k = -6
k= 2
P (|X – 1

Example 1.1. A
throws 6 before 1
begins, show that
Solution
Given that A
Total number
[(1, 5), (2, 4),
A wins if he t
The probabili
The probabili

$$P(2 < X < 14) \geq \frac{3}{4}$$

1.22.2 Example: A random variable X has a mean 12, variance 9, with an unknown probability distribution, find P (|X - 12| $\geq$ 6)

Solution

$$P(\,|X - 12| \geq 6\,) = 1 - P(\,|X - 12| < 6)$$

$$= 1 - P(-6 < |X - 12| < 6)$$

$$= 1 - P(6 < X < 18)$$

$\mu - k\sigma = 6$	$\mu + k\sigma = 18$
$12 - k(3) = 6$	$12 + k(3) = 18$
$-3k = -6$	$3k = 6$
$k = 2$	$k = 2$

$$P(|X - 12| \geq 6) = 1 - P[\mu - k\sigma < X < \mu + k\sigma]$$

$$= 1 - P[12-(2)(3) < X < 12 + (2)\,3]$$

$$= 1 - [\geq 1 - \frac{1}{k^2}] = 1 - \left[\geq 1 - \frac{1}{4}\right]$$

$$= 1 - \left[\geq \frac{3}{4}\right] \leq \frac{1}{4}$$

SOLVED EXAMPLES

Example 1.1. A and B throw alternately with a pair of ordinary dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61. **(Reg. April/May 2004 Set 3)**

Solution

Given that A and B throw alternately with a pair of ordinary dice.

Total number of out comes = 36

[(1, 5), (2, 4), (3, 3), (4, 2) (5,1)]

A wins if he throws 6 before B throws 7.

The probability of getting 6 is $= \dfrac{5}{36}$ .... (1)

The probability of not getting 6 is $\left(1 - \dfrac{5}{36}\right) = \dfrac{31}{36}$

$$[(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)]$$

The probability of getting 7 is $= \dfrac{6}{36} = \dfrac{1}{6}$

The probability of not getting 7 is $\left(1 - \dfrac{6}{36}\right) = \dfrac{30}{36} = \dfrac{5}{6}$

The chance of winning of B if A fails in first throw (A fails) (B wins) $\dfrac{31}{36} \times \dfrac{1}{6}$

Similarly A will get the second chance if A and B fail in first throw

The probability of A to win in the second throw is = (A fails) (B fails) (A wins)

$$\left(\frac{31}{36}\right)\left(\frac{5}{6}\right)\left(\frac{5}{36}\right) \quad \ldots\ldots\ldots\ldots (2)$$

The probability of B to win in the second throw is =

(A fails) (B fails) (A fails) (B wins)

$$\left(\frac{31}{36}\right)\left(\frac{5}{6}\right)\left(\frac{5}{36}\right)\left(\frac{1}{6}\right)$$

The probability of A to win in the third throw is =

(A fails) (B fails) (A fails) (B fails) (A wins)

$$= \left(\frac{31}{36}\right)\left(\frac{5}{6}\right)\left(\frac{31}{36}\right)\left(\frac{5}{6}\right)\left(\frac{5}{36}\right)$$

The chance of wining of A is

$$= \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \ldots\ldots\ldots$$

$$= \frac{5}{36}\left[1 + \frac{31}{36} \times \frac{5}{6} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 + \ldots\ldots\ldots\right]$$

$$= \frac{5}{36}\left[\frac{1}{1 - \frac{31 \times 5}{36 \times 6}}\right]$$

$$= \frac{5}{36} \times \left[\frac{36 \times 6}{216 - 155}\right]$$

$$= \frac{30}{61}$$

Ans.

Example 1.2. A and B throw alternately with a pair of dice. One who first throws a total of nine wins. What are their respective chances of winning if A starts the game? **(Reg. April/May 2004 Set 2)**

Solution

Given that A and B throw alternatively with a pair of dice.

The probability of getting total of nine is

$$(6, 3)\ (5, 4)\ (4, 5)\ (3, 6) = \frac{4}{36} = \frac{1}{9}$$

Given that A starts the same

The probability of wining the A is $\frac{1}{9} = \left(\frac{1}{9}\right)$

Suppose he can't. Then B will throw

The probability of getting total of nine by B is

$$= \text{(not getting A) (getting B)}$$

$$= \left(1 - \frac{1}{9}\right)\left(\frac{1}{9}\right)$$

$$= \left(\frac{8}{9}\right)\left(\frac{1}{9}\right) = \left(\frac{8}{9}\right)\left(\frac{1}{9}\right)$$

If not the chance will go to given A. If he can get total of nine the probability will be

$$= \text{(not winning A) (not winning B) (winning A)}$$

$$= \left(\frac{8}{9}\right)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right) = \left(\frac{8}{9}\right)^2\left(\frac{1}{9}\right)$$

The probability of the wining B in the second trial

$$= \text{(not wining A) (not wining B) (not wining A) (wining B)}$$

$$= \left(\frac{8}{9}\right)\left(\frac{8}{9}\right)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right)$$

The probability of A wining

$$= \text{(wining A)} + \text{(not wining A) (not wining B) (wining A)}$$

$$= \left(\frac{1}{9}\right) + \left(\frac{8}{9}\right)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right) + \left(\frac{8}{9}\right)\left(\frac{8}{9}\right)\left(\frac{8}{9}\right)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right) + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{9}\left[1 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^4 + \dots\dots\dots\right]$$

$$= \frac{1}{9} \cdot \left[\frac{1}{1 - \left(\frac{8}{9}\right)^2}\right]$$

$$= \frac{1}{9}\left[\frac{1}{1 - \frac{64}{81}}\right]$$

$$= \frac{1}{9} \times \frac{81}{81 - 64}$$

$$= \frac{9}{17}$$

$$= 0.529$$ **Ans.**

Example 1.3. Suppose 5 men out of 100 and 25 woman out of 10,000 are color blind. A color blind person is chosen at random. What is the probability of the person being a male? (Assume male and female to be in equal numbers).

(Reg. April/May 2004 Set3)

Solution

Given that

5 men out of 100 and 25 women out of 10,000 are color blind.

Let x = be color blind person

$$P\left(\frac{X}{M}\right) = \frac{5}{100} = 0.05$$

$$P\left(\frac{X}{F}\right) = \frac{25}{10{,}000} = 0.0025$$

$$P(X) = P(M)\, P\left(\frac{X}{M}\right) + P(F)\, P\left(\frac{X}{F}\right)$$

$$= 0.5 \times 0.05 + 0.5 \times 0.0025$$

$$= 0.025 + 0.00125$$

Example 1.4. A
being used only

Solution

The probabil
A ten digit n
It can be arra
But 0 can no
Thus the tota
Number, is f
Are $\angle 10$ -

We have to fi
Any number
i.e. last two d
04, 08, 12, 1
96
The number
The number
The number
(0 can not cor
The number o
The number o

$$= 0.02625$$

$$P\left(\frac{M}{X}\right) = \frac{P(M)P\left(\frac{X}{M}\right)}{P(M)\,P\left(\frac{X}{M}\right) + P(F)\,P\left(\frac{X}{F}\right)}$$

$$= \frac{(0.5)\times(0.05)}{0.02625}$$

$$= \frac{0.025}{0.02625}$$

$$=0.9523$$

Ans.

Example 1.4. A ten digit number is formed using the digits from 0 – 9, every digit being used only once. Find the probability that the number is divisible by 4.

(Reg. April/May 2004 Set 1)

Solution

The probability that the number is divisible by 4 =?

A ten digit number is formed using the digits from 0 – 9.

It can be arranged in $\angle 10$ ways

But 0 can not come in the first place

Thus the total number of ten digit number

Number, is formed using the digits from 0 – 9

Are $\angle 10 - \angle 9 = 3628800 - 362880$

$= 3265920$

We have to find the probability that the number is divisible by 4

Any number will be divisible by 4 if the last two digits are divisible by 4.

i.e. last two digits could be

04, 08, 12, 16, 20, 24, 28, 32, 36, 40, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 92, 96

The number of 10 digit number ending with 04 are $\angle 8 = 40320$.........(1)

The number of 10 digit number ending with 08 are $\angle 8 = 40320$..........(2)

The number of 10 digit number ending with 12 are $\angle 8 - \angle 7$

(0 can not come on the first place) $= 40320 - 5040 = 35280$

The number of 10 digit number ending with 16 are $= \angle 8 - \angle 7 = 35280$

The number of 10 digit number ending with 20 are $\angle 8 = 40320$

The number of 10 digit number ending with 24 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 28 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 32 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 36 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 40 are $\angle 8 = 40320$
The number of 10 digit number ending with 48 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 52 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 56 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 60 are $\angle 8 = 40320$
The number of 10 digit number ending with 64 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 68 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 72 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 76 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 80 are $\angle 8 = 40320$
The number of 10 digit number ending with 84 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 92 are $\angle 8 - \angle 7 = 35280$
The number of 10 digit number ending with 96 are $\angle 8 - \angle 7 = 35280$

Thus the total number of 10 digit numbers divisible by 4 are

$$= 6 \times 40320 + 16 \times 35280$$
$$= 241920 + 564480$$
$$= 806400$$

The probability that the number is divisible by 4

$$= \frac{806400}{3265920}$$
$$= 0.2469135$$

Ans.

Example 1.5. Cards are dealt one by one from a well shuffled pack until an ace appears. Find the probability that exactly n cards are dealt before the ace appears.

(Reg. April/May 2004 Set4)

Solution

The probability of getting an ace $= \frac{4}{52} = \frac{1}{13}$

Probability of not getting an ace $= 1 - \frac{1}{13}$

The probability

The probability

The probability

= (not

$= \frac{12}{13} \times$

The probability

$\left(\frac{12}{13}\right)^n$

Example 1.6. In a
15% have both brov
town.

1. If he has b
2. If he has brown hair

Solution

1. Probability

$P\left(\frac{Brown\ Eye}{Brown\ Ha}\right.$

2. Probability

$P\left(\frac{Does\ not}{Br}\right.$

$$= \frac{12}{13}$$

The probability of getting an ace in the first card (n = 1) = $\frac{1}{13}$

The probability of getting an ace in the second card (n = 2)

= (not getting in first card) (getting in second card)

$$= \frac{12}{13} \times \frac{1}{13}$$

The probability of getting an ace in the third card (n = 3)

= (not getting in first) (not getting in second) (getting in third)

$$= \frac{12}{13} \times \frac{12}{13} \times \frac{1}{13}$$

The probability of getting an ace in the n^{th} card

$$\left(\frac{12}{13}\right)^{n-1} \times \frac{1}{13}$$

Example 1.6. In a certain town 40% have brown hair, 25% have brown eyes and 15% have both brown hair and brown eyes. A person is selected at random from the town. **(Reg. Nov. 2006)**

1. If he has brown hair, what is the probability that he has brown eyes also?
2. If he has brown eyes, determine the probability that he does not have brown hair.

Solution

1. Probability that person has brown hair and brown eyes also is

$$P\left(\frac{Brown\ Eyes}{Brown\ \ Hair}\right) = P\left(\frac{BE}{BH}\right) = \frac{P(BE \cap BH)}{P(BH)}$$

$$= \frac{15}{40}$$

$$= \frac{3}{8}$$

Ans.

2. Probability that person has brown eyes and does not have brown hair

$$P\left(\frac{Does\ not\ have\ brown\ hair}{Brown\ eyes}\right) = P\left(\frac{BH^C}{BE}\right) = \frac{P\left(BE \cap BH^C\right)}{P(BE)}$$

$$= \frac{10}{25}$$

$$= \frac{2}{5} \qquad \textbf{Ans.}$$

Example 1.7. In a certain college 25% of the students failed in mathematics 15% failed in chemistry and 10% in mathematics and chemistry. A student is selected at random.

1. If he/she failed in chemistry, what is the probability that he/she failed in mathematics.
2. If he/she failed in mathematics, what is the probability that he/she failed in chemistry.

(Reg. Nov. 2006 Set2)

Solution

Given that

25% of the students failed in mathematics

$$P(A) = 0.25$$

15% of the students failed in chemistry

$$P(B) = 0.15$$

10% of the students failed in mathematics and chemistry

$$P(A \cap B) = 0.10$$

1. P(m/c), the probability that the students failed mathematics, given that he/she failed chemistry

$$P(m/c) = \frac{P(m \cap c)}{P(c)} = \frac{0.10}{0.15} = \frac{10}{15} = \frac{2}{3} \qquad \textbf{Ans.}$$

2. P(c/m) the probability that the students failed chemistry, given that he or she failed mathematics

$$P(c/m) = \frac{P(m \cap c)}{P(m)} = \frac{0.10}{0.25} = \frac{10}{25} = \frac{2}{5} \qquad \textbf{Ans.}$$

Example 1.8. Three machines produce 70%, 20% and 10% of the total number of a factory. The percentages of defective output of these machines are 4%, 3% and 2% respectively. An item is selected at random and found defective. Find the probability that it is from **(Reg. Nov. 2006 Set 4), (Supple. Feb. 2007)**

1. Machine – I
2. Machine – II
3. Machine – III

Solution

Machine – I –

Machine – II –

Machine – III

E is the event t

1. The proba

$P(m_1 / E) =$

2. The proba

$P(m_2 / E)$

3. The probat

$P(m_3 / E)$

Solution

Machine – I – m_1 P (m_1) = 0.70
Machine – II – m_2 P (m_2) = 0.20
Machine – III – m_3 P (m_3) = 0.10

E is the event that output is defective. Then

$$P(E/M_1)=0.04$$
$$P(E/M_2)=0.03$$
$$P(E/M3)=0.02$$

1. The probability that defective item is from machine – I – M_1

$$P(m_1/E) = \frac{P(E/m_1)P(m_1)}{P(E/m_1)P(m_1)+P(E/m_2)\,P(m_2)+P(E/m_3)P(m_3)}$$

$$=\frac{(0.04)(0.70)}{(0.04)(0.70)+(0.03)(0.20)+(0.02)(0.10)}$$

$$=\frac{0.028}{0.028+0.006+0.002}$$

$$=\frac{0.028}{0.036}=0.777$$ **Ans.**

2. The probability that defective item is from machine – II – m_2

$$P(m_2/E)=\frac{P(E/m_2)\,P(m_2)}{P(\cancel{E/m}_1)P(m_1)+P(E/m_2)P(m_2)+P(E/m_3)P(m_3)}$$

$$=\frac{(0.03)(0.20)}{(0.04)(0.70)+(0.03)(0.20)+(0.02)(0.10)}$$

$$=\frac{0.006}{0.036}=0.1666$$ **Ans.**

3. The probability that defective item is from machine – III – m_3

$$P(m_3/E)=\frac{P(E/m_3)\,P(m_3)}{P(E/m_1)P(m_1)+P(E/m_2)P(m_2)+P(E/m_3)P(m_3)}$$

$$=\frac{(0.02)(0.10)}{0.036}=\frac{0.002}{0.036}$$

$$=0.0555$$ **Ans.**

Example 1.9. Two aero planes bomb a target in succession. The probability of each correctly scoring a hit is 0.3 and 0.2 respectively. The second will bomb only if the first misses the target. Find the probability that

1. Target is hit
2. Both fails to score hits

(Supple. Nov. /Dec. 2004 Set3)
(Supple. Feb. 2007 Set1)

Solution

Given that

Two aero planes bomb a target in succession.

Suppose AB_1 and AB_2

$P(AB_1) = 0.3$

$P(AB_2) = 0.2$

The second will bomb only if the first misses the target

1. P (target is hit) $= P(AB_1 \text{ hits})$ or $P(AB_1 \text{ fails and } AB_2 \text{ hits})$

$$= P(AB_1 \text{ hits}) + P(AB_1 \text{ fails and } AB_2 \text{ hits})$$
$$= P(AB_1 \text{ hits}) + P(AB_1 \text{ fails}) \times P(AB_2 \text{ hits})$$
$$= 0.3 + (1 - 0.3)(0.2)$$
$$= (0.3) + (0.7)(0.2)$$
$$= 0.3 + 0.14$$
$$= 0.44$$ **Ans.**

2. P(both fails to score hits) $= P(AB_1 \text{ fails and } AB_2 \text{ fails})$

$$= P(AB_1 \text{ fails}).\ P(AB_2 \text{ fails})$$
$$= (1 - 0.3)(1 - 0.2)$$
$$= (0.7)(0.8)$$
$$= 0.56$$ **Ans.**

Example 1.10. Determine

1) $P(B/A)$

2) $P(A/B^C)$ if A and B are events with

$$P(A) = \frac{1}{3},\ P(B) = \frac{1}{4},\ P(A \cup B) = \frac{1}{2}$$

(Supple. Nov. /Dec. 2004 Set 3)
(Supple. Feb. 2010 Set 2)

Solution

$$P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)}$$

$$P\left(\frac{A}{B^C}\right)=\frac{P(A\cap B^C)}{P(B^C)}$$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\frac{1}{2}=\frac{1}{3}+\frac{1}{4}-P(A\cap B)$$

$$\frac{1}{2}=\frac{4+3}{12}-P(A\cap B)$$

$$\frac{1}{2}=\frac{7}{12}-P(A\cap B)$$

$$P(A\cap B)=\frac{7}{12}-\frac{1}{2}$$

$$=\frac{7-6}{12}$$

$$P(A\cap B)=\frac{1}{12}$$

$$P(A)=P(A\cap B^C)+P(A\cap B)$$

$$\frac{1}{3}=P(A\cap B^C)+\frac{1}{12}$$

$$\frac{4-1}{12}=P(A\cap B^C)$$

$$\frac{1}{4}=P(A\cap B^C)$$

1. $P(B/A)=\dfrac{P(B\cap A)}{P(A)}$

$$=\frac{1/12}{1/3}$$

$$=\frac{1}{12}\times\frac{3}{1}\quad=\frac{1}{4}$$ **Ans.**

2. $P(A/B^C) = \dfrac{P(A \cap B^C)}{P(B^C)} = \dfrac{1/4}{1-\dfrac{1}{4}}$

$$= \frac{1}{4} \times \frac{4}{3} = \frac{1}{3} \qquad \textbf{Ans.}$$

Example 1.11. A class has 10 boys and 5 girls. Three students are selected at random one after the other. Find the probability that.

(Supple. Nov. / Dec.2004 Set 4)

1. First two are boys and third is girl.
2. First and third of same sex and second is of opposite sex.

Solution

Given that

A class has 10 boys, 5 girls

B – 10

G – 5

Three students are selected at random one after the other.

1. First two are boys and third is girl

$$P(B_1 B_2 G_3) = P(B_1)P(B_2/B_1)P(G_3/B_1B_2)$$

$$= \frac{10}{15} \times \frac{9}{14} \times \frac{5}{13}$$

$$= \frac{450}{2730} = \frac{45}{273}$$

$$= \frac{15}{91} \qquad \textbf{Ans.}$$

2. First and third of same sex and second is of opposite sex.

P (boy, girl, boy) or (girl, boy, girl)

$$= P(B_1G_2B_3) + P(G_1B_2G_3)$$

Both events are mutually exclusive

$$= P(B_1)P\left(\frac{G_2}{B_1}\right)P\left(\frac{B_3}{B_1G_2}\right) + P(G_1)P\left(\frac{B_2}{G_1}\right)P\left(\frac{G_3}{G_1B_2}\right)$$

$$= \frac{10}{15} \times \frac{5}{14} \times \frac{9}{13} + \frac{5}{15} \times \frac{10}{14} \times \frac{4}{13}$$

Example 1.12. In

mathematics. The g

a. What is the pr

b. If a student is

the probability

c. A boy?

Solution

$10\% = .10$ gir

$25\% = .25$ bo

$60\% = .60$ gir

$40\% = .40$ bo

a. $P(m) = P(G)$.

$= (.60).($

$= .0600+$

$= .16$

b. $P(G/M) =$

c. $P(B/M) =$

$$= \frac{450}{2730} + \frac{200}{2730} = \frac{650}{2730} = \frac{65}{273}$$

$$= \frac{5}{21}$$ **Ans.**

Example 1.12. In a certain college 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the student body.

(Supple. Nov. / Dec. 2004 Set2)

a. What is the probability that mathematics is being studied?

b. If a student is selected at random and is found to be studying mathematics find the probability that the student is a girl.

c. A boy?

Solution

$$\left.\begin{array}{l} 10\% = .10 \text{ girls} \\ 25\% = .25 \text{ boys} \end{array}\right\} \text{Mathematics}$$

$$\left.\begin{array}{l} 60\% = .60 \text{ girls} \\ 40\% = .40 \text{ boys} \end{array}\right\} \text{student body}$$

a. $$P(m) = P(G).P(M/G) + P(B).P(M/B)$$
$$= (.60).(.10) + (.40) \times (.25)$$
$$= .0600 + .1000$$
$$= .16$$ **Ans.**

b. $$P(G/M) = \frac{P(M \cap G)}{P(G)P(M/G) + P(B).P(M/B)}$$
$$= \frac{P(G).P(M/G)}{P(G).P(M/G) + P(B).P(M/B)}$$
$$= \frac{(.60)(.10)}{(.60)(.10) + (.40)(.25)} = \frac{3}{8}$$

c. $$P(B/M) = \frac{P(B).P(M/B)}{P(G).P(M/G) + P(B).P(M/B)}$$
$$= \frac{(.40)(.25)}{(.60)(.10) + (.40)(.25)} = \frac{5}{8}$$ **Ans.**

Example 1.13. There are three boxes. Box I contains 10 light bulbs of which 4 are defective. Box II Contains 6 light bulbs of which one is defective. Box III contain 8 light bulbs of which 3 are defective. A box is chosen and a bulb is drawn. Find the probability that the bulb is non defective. **(Supple. Feb. 2007 Set1)**

Solution

Given that

	Box I	**Box II**	**Box III**
Light bulbs	10	6	8
Defective	4	1	3

A box is chosen and a bulb is drawn

The probability that the box is chosen

$B_1 \rightarrow$ box one, $B_2 \rightarrow$ box II, $B_3 \rightarrow$ box III

$$P(B_1) = \frac{1}{3}$$

$$P(B_2) = \frac{1}{3}$$

$$P(B_3) = \frac{1}{3}$$

D is the probability of defective bulb

$$P(D) = P(B_1).P(D/B_1) + P(B_2).P(D/B_2) + P(B_3).P(D/B_3)$$

$$= \frac{1}{3}\cdot\frac{4}{10} + \frac{1}{3}\frac{1}{6} + \frac{1}{3}\cdot\frac{3}{8}$$

$$= \frac{4}{30} + \frac{1}{18} + \frac{3}{24}$$

$$= \frac{2}{15} + \frac{1}{18} + \frac{1}{8}$$

$$= \frac{288+120+270}{2160}$$

$$= \frac{678}{2160} = \frac{113}{360}$$

Probability that bulb is non defective

$$= 1 - \frac{113}{360} = \frac{360-113}{360} = \frac{247}{360} = 0.686 \qquad \textbf{Ans.}$$

Example 1.14.
50% of the total
at random and 1
from

Solution

E is the eve

1. The probab

$P(M_A/E$

2. The probab

$P(M_B/E$

Example 1.14. In a bolt factory machines A, B, C manufacture 20%, 30%, and 50% of the total of their output and 6%, 3% and 2% are defective. A bolt is drawn at random and found to be defective. Find the probability that it is manufactured from

1. Machine A
2. Machine B
3. Machine C

(Supple. Feb. 2007 Set 2)
(Reg. Nov. 2006 Set 4)

Solution

Machine A - M_A	$P(M_A) = 0.20$
Machine B - M_B	$P(M_B) = 0.30$
Machine C - M_C	$P(M_C) = 0.50$

E is the event that bolt is defective then

$$P(E/M_A) = 0.06$$
$$P(E/M_B) = 0.03$$
$$P(E/M_C) = 0.02$$

1. The probability that defective bolt is from machine A - M_A

$$P(M_A/E) = \frac{P(E/M_A)\,P(M_A)}{P(E/M_A)P(M_A) + P(E/M_B)\,P(M_B) + P(E/M_C)\,P(M_C)}$$

$$= \frac{(0.06)\,(0.20)}{(0.06)\,(0.20) + (0.03)\,(0.30) + (0.02)\,(0.50)}$$

$$= \frac{0.012}{0.012 + 0.009 + 0.01} = \frac{0.012}{0.031} = 0.3870$$

2. The probability that defective bolt is from machine B - M_B

$$P(M_B/E) = \frac{P(E/M_B)\,P(M_B)}{P(E/M_A)P(M_A) + P(E/M_B)\,P(M_B) + P(E/M_C)\,P(M_C)}$$

$$= \frac{(0.03)\,(0.30)}{(0.06)\,(0.20) + (0.03)\,(0.30) + (0.02)\,(0.50)}$$

$$= \frac{0.009}{0.012 + 0.009 + 0.01} = \frac{0.009}{0.031} = 0.2903$$

3. The probability that defective bolt is from machine C - M_C

$$P(M_C/E)=\frac{P(E/M_C)\,P(M_C)}{P(E/M_A)P(M_A)+P(E/M_B)\,P(M_B)+P(E/M_C)\,P(M_C)}$$

$$=\frac{(.02)(.50)}{(.06)(.20)+(.03)(.30)+(.02)(.50)}$$

$$=\frac{.01}{.012+.009+.01}=\frac{.01}{.031}=0.3225$$

Example 1.15. Box A contains 5 red and 3 white marbles and box B contains 2 red and 6 white marbles.If a marble is drawn from each box, what is the probability that they are both of the same color? **(Supple. Feb. 2007 Set 4)**

Solution **Given that**

	Marbles		
	Red	**White**	**Total**
Box A	5	3	8
Box B	2	6	8
Total	7	9	

Probability of the red marble from box A, $P(R_A)=\frac{5}{8}$

Probability of the red marble from box $B, P(R_B)=\frac{2}{8}$

Probability of the red marble from box A and box B = $P(R_A)\times P(R_B)$

$$=\frac{5}{8}\times\frac{2}{8}=\frac{10}{64}=\frac{5}{32}$$

Probability of the white marble from box A, $P(W_A)=\frac{3}{8}$

Probability of the white marble from box $B, A(W_B)=\frac{6}{8}$

Probability of the white marble from box A and box B

$P(W_A \text{ and } W_B) = P(W_A)\times P(W_B)$

$$=\frac{3}{8}\times\frac{6}{8}=\frac{18}{64}=\frac{9}{32}$$

The probability
same color

Example 1.16.
30 white, 20 blu
drawing. Find th

1. Both are wh
2. First is red a

Solution

Given that

Two marble
20 blue, 15 orang

1. Both are whi

Total numbe

P (first marb

After replace

P (second ma

P (both are w

2. First is red ar

P (first marbl

P (second ma

P (first is red

The probability of a marble which is drawn from each box, they are both of the same color

$$P(R_A \text{ and } R_B) \text{ or } P(W_A \text{ and } W_B)$$

$$= \frac{5}{32} + \frac{9}{32} = \frac{14}{32} = \frac{7}{16} \qquad \textbf{Ans.}$$

Example 1.16. Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue and 15 orange marbles, with replacement being made after each drawing. Find the probability that

(Reg. Apr. /May 2005 Set 3)
(Supple. Nov. /Dec. 2005 Set 2)
(Supple. Nov.2008 Set 4)

1. Both are white
2. First is red and second is white.

Solution

Given that

Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue, 15 orange marbles. With replacement being made after each drawing

1. Both are white

   Total number of marbles = 10 + 30 + 20 + 15 = 75

   $$P\text{ (first marble drawn is white)} = \frac{30}{75}$$

   After replacement

   $$P\text{ (second marble drawn is white)} = \frac{30}{75}$$

   $$P\text{ (both are white)} = \frac{30}{75} \times \frac{30}{75} = \frac{4}{25} \qquad \textbf{Ans.}$$

2. First is red and second is white

   $$P\text{ (first marble drawn is red)} = \frac{10}{75}$$

   $$P\text{ (second marble drawn is white)} = \frac{30}{75}$$

   $$P\text{ (first is red and second is white)} = \frac{10}{75} \times \frac{30}{75}$$

$$= \frac{4}{75} \qquad \textbf{Ans.}$$

Example 1.17. A businessman goes to hotels X, Y, Z, 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z. Hotels have faulty plumbing. What is the probability that business mans room having faulty plumbing is assigned to hotel Z.

(Reg. April / May 2005 Set3)
(Supple. Nov. / Dec. 2005 Set3)
(Supple. Nov.2008 Set 4)

Solution

Given that

A businessman goes to hotels X, Y, Z, 20%, 50%, 30% of the time respectively.

$$P(X) = \frac{20}{100} = 0.20$$

$$P(Y) = \frac{50}{100} = 0.50$$

$$P(Z) = \frac{30}{100} = 0.30$$

And 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbing

FP $\rightarrow$ faulty plumbing

$$P(FP/X) = \frac{5}{100} = 0.05$$

$$P(FP/Y) = \frac{4}{100} = 0.04$$

$$P(FP/Z) = \frac{8}{100} = 0.08$$

That is the probability that business mans room being faulty plumbing is assigned to hotel Z.

$$P(Z/FP) = \frac{P(Z) \, . \, P(FP/Z)}{P(X).P(FP/X) + P(Y) \, . \, P(FP/Y) + P(Z) \, . \, P(FP/Z)}$$

$$= \frac{(0.30) \times (0.08)}{(0.20) \times (0.05) + (0.50) \times (0.04) + (0.30) \times (0.08)}$$

Example 1.18.
respectively. It i
are defective.

1. What is the
2. If a car purc
   produced by

Solution

Given that
respectively.

And 2%, 3%

1. What is
   $P(D) =$
   $= ($
   $= ($
   P (D) =

2. This defe

$$= \frac{0.024}{0.01+0.02+0.024}$$

$$= \frac{0.024}{0.054} = 0.4444$$

Ans.

Example 1.18. Companies B_1, B_2, B_3, produce 30%, 45%, 25%, of the cars respectively. It is known that 2%, 3%, 2% of these cars produced from B_1, B_2, B_3 are defective.

1. What is the probability that a car purchased is defective.
2. If a car purchased is found to be defective what is the probability that this car is produced by the company B.

(Supple. Nov. /Dec. 2005 Set2)

Solution

Given that companies B_1, B_2, B_3 produce 30%, 45%, 25% of the cars respectively.

$$P(B_1) = \frac{30}{100} = 0.30$$

$$P(B_2) = \frac{45}{100} = 0.45$$

$$P(B_3) = \frac{25}{100} = 0.25$$

And 2%, 3% and 2% of these cars produced from B_1, B_2, B_3 are defective

$$P(D/B_1) = \frac{2}{100} = .02$$

$$P(D/B_2) = \frac{3}{100} = .03$$

$$P(D/B_3) = \frac{2}{100} = .02$$

1. What is the probability that a car purchased is defective.

$$P(D) = P(B_1).P(D/B_1) + P(B_2).P(D/B_2) + P(B_3)P(D/B_3)$$

$$= (0.30)\,(0.02) + (0.45)\,(0.03) + (0.25)\,(0.02)$$

$$= 0.006 + 0.0135 + 0.27$$

$$P(D) = 0.2895$$

Ans.

2. This defective car is produced by the company B.

$$P(B_1/D)=\frac{P(B_1)P(D/B_1)}{P(B_1)\,P(D/B_1)+P(B_2)\,P(D/B_2)+P(B_3)P(D/B_3)}$$

$$=\frac{0.006}{0.2895} \qquad \textbf{Ans.}$$

Example 1.19. In the experiment of throwing a pair of dice, consider the events

1. Two numbers are equal
2. Sum is ≥ 7

Solution

In the experiment of throwing a pair of dice, total numbers of outcomes are 36.

1. Two numbers are equal

   A = { (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6)}

   total number of favorable cases = 6

		Die-1					
		1	2	3	4	5	6
Die – 2	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	6, 5)	(6, 6)

$$P(A)=\frac{\text{total number of favourable cases}}{\text{Total number of possible cases}}$$

$$=\frac{6}{36}$$

$$=\frac{1}{6}$$

2. Sum is ≥ 7

   For sum is ≥ 7

   B = { (6, 1) (5

   (5, 4) (4, 5) (4

   Total number

Example 1.20. Th
hits the target is 3/5
of them hits the ta
target

Solution

Given that

By the condition

For independent

By the addition r

$$= \frac{1}{6}$$ **Ans.**

2. Sum is ≥ 7

For sum is ≥ 7 may be {7, 8, 9, 10, 11, 12}

B = { (6, 1) (5, 2) (4, 3) (3, 4) (3, 5) (1, 6) (6, 2) (5, 3) (4, 9) (3, 5) (2, 6) (6, 3) (5, 4) (4, 5) (4, 5) (3, 6) (6, 4) (5, 5) (4, 6) (6, 5) (5, 6) (6, 6)}

Total number of favorable cases = 21

$$P(B) = \frac{\text{total number of favourable cases}}{\text{Total number of possible cases}}$$

$$= \frac{21}{36} = \frac{7}{12}$$ **Ans.**

Example 1.20. The probability that A hits a target is 1/3 and the probability that B hits the target is 3/5. Both shoot at the target. Find the probability that at least one of them hits the target, that is, find the probability that A or B (or both) hits the target

Solution

Given that

$$P(A) = \frac{1}{3}$$

$$P(B) = \frac{3}{5}$$

$$P(A \cup B) = ?$$

By the conditional probability

$$P(A \cap B) = P(A).P(B/A)$$

For independent event

$$P(A \cap B) = P(A).\ A(B)$$

$$= \frac{1}{3} \times \frac{3}{5} = \frac{3}{15} = \frac{1}{5}$$

By the addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{3} + \frac{3}{5} - \frac{1}{5}$$

$$= \frac{5+9-3}{15} = \frac{11}{15}$$

$$(A \cup B) = 0.7333$$ **Ans.**

Example 1.21. A and B be events with P (A) = 0.5

P (B) = 0.4,

And P (A ∩ B) = 0.2

Find

1. $P(A/B)$
2. $P(A \cup B)$
3. $P(B^C)$
4. $P(A^C/B^C)$

Solution

Given that P (A) = 0.5

P (B) = 0.4

P (A ∩ B) = 0.2

$$P(A/B) = ?,\ P(A \cup B) = ?,\ P(B^C) = ?$$

We know that by the conditional probability

1. $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.2}{0.4} = \frac{2}{4} = \frac{1}{2} = 0.5$$ **Ans.**

2. by the addition theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.5 + 0.4 - 0.2$$

$$= 0.9 - 0.2$$

$$= 0.7$$ **Ans.**

3. $$P(B^C) = 1 - P(B)$$

$$= 1 - 0.4$$

$$= 0.6$$ **Ans.**

4. $P(A^C/I$

Example 1.22. Fo

probability that the

Solution

Four cards are

They are all ca

Hence the requ

Solution By multip

The probability

Similarly, the p

4. $$P\left(A^C/B^C\right)=\frac{P\left(A^C\cap B^C\right)}{P\left(B^C\right)}$$

$$=\frac{P\left((A\cup B)^C\right)}{1-P(B)}=\frac{1-P(A\cup B)}{1-P(B)}$$

$$=\frac{1-0.7}{1-0.4}=\frac{0.3}{0.6}=\frac{3}{6}=\frac{1}{2}=0.5$$ **Ans.**

Example 1.22. Four cards are drawn at random from a pack of 52 cards. Find the probability that they are all cards of diamonds.

Solution

Four cards are drawn at random from a pack of 52 cards.

$$^{52}C_4=\frac{\angle 52}{\angle 4\ \angle 52-4}=\frac{52\times 51\times 50\times 49}{4\times 3\times 2\times 1}$$

They are all cards of diamonds

$$^{13}C_4=\frac{\angle 13}{\angle 4\ \angle 13-4}=\frac{13\times 12\times 11\times 10}{4\times 3\times 2\times 1}$$

Hence the required probability $=\dfrac{13_{C4}}{52_{C4}}$

$$=\frac{13\times 12\times 11\times 10}{4\times 3\times 2\times 1}\times\frac{4\times 3\times 2\times 1}{52\times 51\times 50\times 49}$$

$$=\frac{11}{4165}=.0026$$

Solution By multiplication theorem

The probability of picking a card of diamond in the

$$\text{First draw}=\frac{13}{52}$$

Similarly, the probability of picking a card of diamond in the second draw

$$=\frac{12}{52}$$

Similarly $\dfrac{11}{50},\dfrac{10}{49}$ respectively

Hence the probability of drawing 4 cards of diamond

$$=\frac{13}{52}\times\frac{12}{51}\times\frac{11}{50}\times\frac{10}{49}=\frac{11}{4165}=0.0026$$

Example 1.23. Three cards are drawn from a pack of 52 cards. Find the probability that they are of the same color.

Solution Three cards can be drawn from the pack in 52_{C_3} ways

$$=\frac{\angle 52}{\angle 3 \angle 52-3}=\frac{\angle 52}{\angle 3 \angle 49}$$

$$=\frac{52\times 51\times 50}{3\times 2}=52\times 17\times 25$$

Three cards can be drawn from the 26 red cards in 26_{C_3} ways

$$=\frac{\angle 26}{\angle 3\ \angle 26-3}=\frac{\angle 26}{\angle 3 \angle 23}$$

$$=\frac{26\times 25\times 24}{3\times 2\times 1}=13\times 25\times 8$$

Hence the probability that the three cards are all red

$$=\frac{26_{C_3}}{52_{C_3}}=\frac{13\times 25\times 8}{52\times 17\times 25}=\frac{2}{17}$$

Similarly, the probability that three cards are all black is also = 2 / 17

Therefore the probability that the three cards are of the same color (either all red or all black)

$$P(A\cup B)=P(A+B)=P(A)+P(B)$$

$$=\frac{2}{17}+\frac{2}{17}=\frac{4}{17}$$ By addition theorem (second method)

By multiplication theorem

The probability of picking one card of same color in first draw $=\dfrac{26}{52}$

The probability of picking second card of same color in second draw $=\dfrac{25}{51}$

The probability of picking third card of same color in third draw $=\dfrac{24}{50}$

Hence the probability of picking three cards of same color = $\dfrac{26}{52}\times\dfrac{25}{51}\times\dfrac{24}{50}=\dfrac{2}{17}$

Similarly the probability that the three cards are all black is also $\dfrac{2}{17}$

Therefore the required probability (either all red or all black)

$$=\frac{2}{17}+\frac{2}{17}=\frac{4}{17} \qquad \textbf{Ans.}$$

Example 1.24. Two cards are drawn from a pack of 52 cards. Find the probability that

1. They are both aces
2. Find the probability if the first card is replaced before the second is drawn

1. Method – I **Solution By multiplication theorem**

The probability of picking an ace in the first draw = $\dfrac{4}{52}=\dfrac{1}{13}$

When an ace has been picked only 3 aces are left in the remaining 51 cards. Hence the probability of picking an ace in the second draw.

$$=\frac{3}{51}=\frac{1}{17}$$

Hence the required probability of drawing two aces

$$=\frac{1}{13}\times\frac{1}{17}=\frac{1}{221} \qquad \textbf{Ans.}$$

Method – II

Required probability is $\dfrac{4_{C_2}}{52_{C_2}}=\dfrac{1}{221}$ **Ans.**

2. Method – I **Solution By multiplication theorem**

If the first card is replaced before the second draw, the probability of picking an ace in the second draw is also 1/13

Hence the required probability = $\dfrac{1}{13}\times\dfrac{1}{13}=\dfrac{1}{169}$ **Ans.**

Method – II Conditional probability

$$P(A\cap B)=P(A)\,.\,P(B/A)$$

$$=\frac{4_{C_1}}{52_{C_1}}\times\frac{4_{C_1}}{52_{C_1}}$$

$$=\frac{1}{13}\times\frac{1}{13}=\frac{1}{169}$$ **Ans.**

Example 1.25. Find the probability of drawing

1. An ace
2 A card of heart
3. An ace or a card of heart from a pack of 52 cards

Solution

1. The probability of drawing an ace $=\frac{4}{52}=\frac{1}{13}$

2. The probability of drawing a heart $=\frac{13}{52}=\frac{1}{4}$

3. The number of favorable cases to get an ace or heart is 16. (since total number of heart cards 13 + total ace 4 = 17 but one of the card is in both an ace and a heart)

   Hence the required probability $=\frac{16}{52}=\frac{4}{13}$ **Ans.**

Example 1.26. Determine the probability P of a red marble appears in the random drawing of 1 marble from a box containing 4 white, 3 red and 5 blue marbles.

Solution

There is total number of marble

$$=4+3+5=12$$

In the random drawing of 1 marble from a box,

Total number of possible out comes are $12_{C_1}=12$

In the random drawing of 1 red marble

Number of favorable cases $3_{C_1}=3$

Required probability $=\frac{3}{12}=\frac{1}{4}$ **Ans.**

Example 1.27. De
an ordinary deck of
1. King
2. Face card (jac
3. Red face card

Solution

1. King, require
2. Face card, req
3. Red face card.

Example 1.28. Dete
die

Solution

Total number
Sample space S = {1
Even number
Event E = {2, 4, 6}

Required prob

Example 1.29. Dete
fair coins.

Solution

Total number c
Sample space S
Total is 8
At least one tai
Event E = {HH

Hence required

Example 1.27. Determine the probability P of the drawing of a single card from an ordinary deck of 52 cards

1. King
2. Face card (jack, queen, or king)
3. Red face card

Solution

1. King, required probability $= \dfrac{4}{52}$ **Ans.**

2. Face card, required probability $= \dfrac{12}{52} = \dfrac{3}{13}$ **Ans.**

3. Red face card, required probability $= \dfrac{6}{52} = \dfrac{3}{26}$ **Ans.**

Example 1.28. Determine the probability P of an even number in the throw of fair die

Solution

Total number of possible out comes in the throwing of fair die = 6

Sample space S = {1, 2, 3, 4, 5, 6}

Even number may be (2, 4, 6)

Event E = {2, 4, 6}

$$\text{Required probability} = \frac{\text{Total number of favourable cases}}{\text{Total number of possible out comes}}$$

$$= \frac{3}{6} = \frac{1}{2}$$ **Ans.**

Example 1.29. Determine the probability P of at least one tail appears in the toss of 3 fair coins.

Solution

Total number of possible out comes in the tossing of 3 fair coins are

Sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Total is 8

At least one tail appears, favorable cases may be

Event E = {HHT, HTH, HTT, THH, THT, TTH, TTT}

$$\text{Hence required probability} = \frac{\text{Total number of favourable cases}}{\text{Total number of possible out comes}}$$

$$= \frac{7}{8} \qquad \textbf{Ans.}$$

Example 1.30. In the experiment of throwing a pair of dice, consider the events

1. a sum 7
2. a sum of 6 occurs

Solution

In the experiment of throwing a pair of dice, total number of possible out comes are 36 as shown in Example 1.19

a sum 7

$$\text{Event A} = \{(6, 1)\,(5, 2)\,(4, 3)\,(3, 4)\,(2, 5)\,(1, 6)\}$$

$$P(A) = \frac{\text{Total number of favourable cases}}{\text{Total number of possible out comes}}$$

$$= \frac{6}{36} = \frac{1}{6} \qquad \textbf{Ans.}$$

1. sum of 6 occurs

$$\text{Event B} = \{(5, 1)\,(4, 2)\,(3, 3)\,(2, 4)\,(1, 5)\}$$

$$P(B) = \frac{\text{Total number of favourable cases}}{\text{Total number of possible out comes}}$$

$$= \frac{5}{36} \qquad \textbf{Ans.}$$

Example 1.31. If probability of an event A is = 0.1

The probability of an event B is = 0.3

The probability of $A \cap B = 0.5$

Are the events are independent.

Solution

Given that

$$P(A) = 0.1,\ P(B) = 0.3,\ P(A \cap B) = 0.5$$

$$P(A).\ P(B) = (0.1) \times (0.3)$$

It is clear that

$$P(A \cap B) \neq P(A).\ P(B)$$

$\Rightarrow$ The events are not independent.

Example 1.32. If

$A \cup B = S$, find P (I

Solution

Given that

We know tha

Example 1.33. If A

1. $P(A^C \cap B)$
2. $P(A) \leq P($

Solution

We know that

Given that $A \subseteq$

By 1. P (B) – P

We know that p

Example 1.32. If $B \subseteq A$ and A and B are two events such that $P(A) = 5P(B)$ and $A \cup B = S$, find $P(B) = ?$

Solution

Given that

$$B \subseteq A,\ A \cup B = S$$

$$\Rightarrow A \cap B = B$$

We know that

$$P(P \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(S) = 1$$

$$1 = 5P(B) + P(B) - P(B)$$

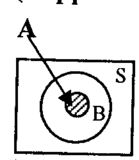
$$\Rightarrow P(B) = \frac{1}{5}$$ **Ans.**

Example 1.33. If $A \subseteq B$ then prove that

1. $P(A^C \cap B) = P(B) - P(A)$
2. $P(A) \leq P(B)$

(Supple. JNTU 2004)

Solution



We know that

$$P(B) = P(A^C \cap B) + P(A \cap B)$$

Given that $A \subseteq B$

$$\Rightarrow A \cap B = A$$

$$P(B) = P(A^C \cap B) + P(A)$$

$$P(A^C \cap B) = P(B) - P(A) \text{ proved}$$

By 1. $P(B) - P(A) = P(A^C \cap B)$

We know that probability of any event is greater than or equal to 0

$$P(A^C \cap B) \geq 0$$

$$P(B) - P(A) \geq 0$$

$$P(B) \geq P(A)$$

$$\text{Or } P(A) \leq P(B) \text{ proved}$$

Example 1.34. If A and B are mutually exclusive events, then prove that $P(A) \le P(B^C)$

(JNTU 2000)

Solution

Given that A and B are mutually exclusive events

$$P(A \cap B) = 0$$

We have

$$P(A \cup B) = P(A) + P(A) - P(A \cap B)$$
$$P(A \cup B) = P(A) + P(B)$$

Probability of any event is ≥ 0 or ≤ 1

$$P(A \cup B) \le 1$$
$$P(A) + P(B) \le 1$$
$$P(A) \le 1 - P(B)$$
$$P(A) \le P(B^C) \text{ proved}$$

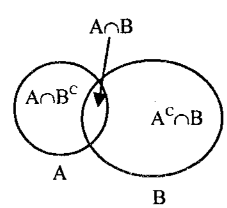
Example 1.35.

If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(A \cap B) = \frac{1}{5}$ then find

(Reg. Nov. 2006 set 4)

1. $P(A \cup B)$
2. $P(A^C \cap B)$
3. $P(A \cap B^C)$
4. $P(A^C \cap B^C)$

Where (A^C = compliment of A)



1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

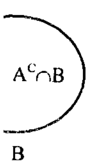
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{5}$$
$$= \frac{15 + 10 - 6}{30}$$

$$P(A) = P(A \cap B^C) + P(A \cap B)$$
$$P(B) = P(A \cap B) + P(A^C \cap B)$$

$$P(A \cup B) = \frac{19}{30} \text{ Ans.}$$

that $P(A) \le P(B^C)$
(JNTU 2000)

Nov. 2006 set 4)



2. $P(A^C \cap B) = ?$

$$A(B) = P(A \cap B) + P(A^C \cap B)$$

$$\frac{1}{3} = \frac{1}{5} + P(A^C \cap B)$$

$$\frac{1}{3} - \frac{1}{5} = P(A^C \cap B)$$

$$\frac{5-3}{15} = P(A^C \cap B)$$

$$\frac{2}{15} = P(A^C \cap B)$$

Ans.

3. $P(A \cap B^C) = ?$

$$P(A) = P(A \cap B^C) + P(A \cap B)$$

$$\frac{1}{2} = P(A \cap B^C) + \frac{1}{5}$$

$$\frac{1}{2} - \frac{1}{5} = P(A \cap B^C)$$

$$\frac{5-2}{10} = P(A \cap B^C)$$

$$\frac{3}{10} = P(A \cap B^C)$$

Ans.

4. $P(A^C \cap B^C) = ?$

$$P(A^C \cap B^C) = P(A \cup B)^C$$

$$= 1 - P(A \cup B)$$

$$= 1 - \frac{19}{30}$$

$$= \frac{30-19}{30}$$

$$P(A^C \cap B^C) = \frac{11}{30}$$

Ans.

Example 1.36. The probabilities that students A, B, C, D solve a problem are $\frac{1}{3}, \frac{2}{5}, \frac{1}{5}$ and $\frac{1}{4}$ respectively. If all of them try to solve the problem, what is the probability that the problem is solved? **(Supple. Feb. 2007 Set 2)**

Solution

Given that the probabilities that students A, B, C, D solve a problem are $\frac{1}{3}, \frac{2}{5}, \frac{1}{5}$ and $\frac{1}{4}$ respectively.

$$P(A) = \frac{1}{3}$$

$$P(B) = \frac{2}{5}$$

$$P(C) = \frac{1}{5}$$

$$P(D) = \frac{1}{4}$$

The probability that the problem is solved

$$= P(A \cup B \cup C \cup D)$$

$$= 1 - P(A \cup B \cup C \cup D)^C \text{ (by De-Morgan's law)}$$

$$= 1 - P(A^C)\, P(B^C)\, P(C^C)\, P(D^C)$$

(All given events A, B, C, D are independent)

$$= 1 - \left(1 - \frac{1}{3}\right)\left(1 - \frac{2}{5}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{4}\right)$$

$$= 1 - \left(\frac{2}{3}\right)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)\left(\frac{3}{4}\right)$$

$$= 1 - \frac{72}{300}$$

$$= \frac{228}{300} = \frac{114}{150} = \frac{57}{75} = \frac{19}{25}$$

$$= \frac{19}{25}$$ **Ans.**

Example 1.37.
What is the prot
Solution
Given tha
An intege
They are |
Total nur
The numt
A = [7, 14
140, 147, 154, 1

The numb
B = [9, 18
162, 171, 180, 1

P (A∪B) =
A∩B = {6

Example 1.38. I
1. P(A^C)
2. P(B) :

Solution **Proof**
1. We knc
S = A
For the
P(S) = 1

Example 1.37. An integer is chosen at random from the first 200 positive integers. What is the probability that the integer chosen is divisible by 7 or 9?

Solution

Given that

An integer is chosen at random from the first 200 positive integers.

They are [1, 2, 3, 4, ...200]

Total numbers of possible cases are 200

The numbers are divisible by 7

A = [7, 14, 21, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105, 112, 119, 126, 133, 140, 147, 154, 161, 168, 175, 182, 189, 196]

$$\text{Total } 28,\ P(A) = \frac{28}{200}$$

The numbers are divisible by 9

B = [9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, 144, 153, 162, 171, 180, 189, 198]

$$\text{Total } 22,\ P(B) = \frac{22}{200}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$A \cap B = \{63, 126, 189\}$

$$P(A \cap B) = \frac{3}{200}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{28}{200} + \frac{22}{200} - \frac{3}{200}$$

$$= \frac{28 + 22.3}{200} = \frac{50 - 3}{200} = \frac{47}{200}$$

Example 1.38. Prove that

1. $P(A^C) = 1 - P(A) \leq 1$
2. $P(B) \leq P(A)$ when $B \subset A$ **(Reg. April,/May 2005 Set 4)**

Solution Proof

1. We know that

$S = A \cup A^C$ .....(1) Where A and A^C are disjoint.

For the certain event S, we have P(S) =1 from the equation (1)

$P(S) = P(A \cup A^C)$

$1 = P(A \cup A^C)$

$1 = P(A) + P(A^C)$

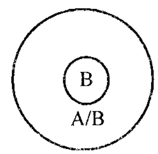
By adding [–P(A)] to both sides gives us

$1 - P(A) = P(A) + P(A^C) - P(A)$

$\boxed{1 - P(A) = P(A^C)}$ Proved

2. $P(B) \leq P(A)$ when $B \subseteq A$

Given that $B \subseteq A$



$$A = B \cup (A/B)$$

Where B and A/B are disjoint

Where have $P(A/B) \geq 0$

Hence

$\boxed{P(B) \leq P(A)}$ Proved

Example 1.39. A box contains 2 white balls and 2 blue balls. Two balls are drawn at random. Find the probability P they are same color.

Solution

Given that box contains 2 white balls and 2 blue balls

Two balls are drawn at random

Total number of ball $2 + 2 = 4$

Total number of cases to draw two balls $= 4_{C_2} = \frac{\angle 4}{\angle 2 \angle 2} = \frac{4 \times 3}{4} = 6$

Number of favorable cases of the same color balls may be 2 white or 2 blue balls

$$2_{C_2} = 1$$

Probability of the 2 white balls $P(A) = \frac{1}{6}$

Similarly probability of the 2 blue balls $P(B) = \frac{1}{6}$

Required probability P, are

Same color $= P(A) + P(B)$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$ **Ans.**

Example 1.40.

Find the probabi

1. The two ca

2. The two ca

Solution

Given that

Two cards are se

The probability t

1. The two ca

Total numl

There are five eve

If selected both n

Means number of

If selected l

The sum wi

Means num

Total numb

Required pr

2. The two car

Total numb

Numbers of

Example 1.40. Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability that the sum is even if

(Reg. Nov. 2006 Set3)

(Supple. Nov. 2008)

1. The two cards are drawn together.
2. The two cards are drawn on after the other with replacement.

Solution

Given that

Two cards are selected at random from 10 cards numbered 1 to 10

The probability that the sum is even number =?

1. The two cards are drawn together

$$\text{Total number of possible cases } 10_{C_2} = \frac{\angle 10}{\angle 2\ \angle 10-2}$$

$$= \frac{10 \times 9 \times \angle 8}{2 \times 1 \times \angle 8}$$

There are five even and five odd numbered cards = 45

If selected both numbers are even number then sum will be even no.

$$\text{Means number of favorable cases } = 5_{C_2} = \frac{\angle 5}{\angle 2\ \angle 5-2}$$

$$= \frac{5 \times 4 \angle 3}{2\ \angle 3}$$

$$= 10$$

If selected both numbers are odd number

The sum will be even number

Means number of favorable cases = $5_{C_2} = 10$

Total number of favorable cases = 10 + 10 = 20

Required probability = $\frac{20}{45} = \frac{4}{9}$ **Ans.**

2. The two cards are drawn on after the other with replacement

Total number of possible cases $10_{C_1} \times 10_{C_1} = 10 \times 10 = 100$

Numbers of favourable cases, both are even = $5_{C_1} \times 5_{C_1} = 25$

Numbers of favorable cases, both are odd $= 5_{C_1} \times 5_{C_1} = 25$

Total number of favorable cases = 25 + 25 = 50

$$\text{Required probability} = \frac{50}{100}$$

$$= \frac{1}{2} \qquad \textbf{Ans.}$$

Example 1.41. Suppose A and B are two events with P (A) = 0.7, P (B) = 0.2 and $P(A \cap B) = 0.1$. Find the probability that

a. A does not occur

b. B does not occur

c. A or B occur

d. Neither A nor B occurs

Solution

a. Probability of A does not occur

$$P(A^C) = 1 - P(A)$$
$$= 1 - 0.7 = 0.3 \qquad \textbf{Ans.}$$

b. Probability of B does not occur

$$P(B^C) = 1 - P(B)$$
$$= 1 - 0.2 = 0.8 \qquad \textbf{Ans.}$$

c. Probability of A or B occur

$$P(A \text{ or } B) = P(A \cup B)$$
$$= P(A) + P(B) - P(A \cap B)$$
$$= 0.7 + 0.2 - 0.1$$
$$= 0.9 - 0.1$$
$$= 0.8 \qquad \textbf{Ans.}$$

d. Probability that neither A nor B occurs is the complement of $(A \cup B)$

$$P(\text{neither A nor B}) = P((A \cup B)^C)$$
$$= 1 - P(A \cup B)$$
$$= 1 - 0.8$$
$$= 0.2 \qquad \textbf{Ans.}$$

Example 1.42. Determine the probability in the tossing of pair of fair coins

1. Head on first toss
2. Head on second toss

Solution

In the tossing of pair of fair coins total number of possible out comes are

$$S = \{HH, HT, TH, TT\}$$

S.No.	Coin 1st	Coin 2nd	Probability
1	H	H	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
2	H	T	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
3	T	H	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
4	T	T	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

1. Head on first toss

   Number of favorable cases are E = {HH, HT}

   Required probability $= \frac{2}{4} = \frac{1}{2}$ **Ans.**

2. Head on second toss

   Numbers of favorable cases are {HH, TH}

   Required probability $= \frac{2}{4} = \frac{1}{2}$ **Ans.**

Example 1.43. A number is picked from 1 to 20; both inclusive find the probability that it is prime.

Solution

We have total 20 numbers

i.e. total number of out comes are 20

Number of favorable cases to get a prime number is equal to the total number of prime numbers from 1 to 20

These are {2, 3, 5, 7, 11, 13, 17, and 19}

Which are 8 in number

$$\text{Therefore the required probability} = \frac{\text{Number of favourable cases}}{\text{Total number of out comes}}$$

$$=\frac{8}{20} \qquad \textbf{Ans.}$$

Example 1.44. A number is picked from 1 to 30, both inclusive find the probability that it is prime.

Solution

We have total 30 numbers

i.e. total number of out comes are 30

Number of favorable cases to get a prime number is equal to the total number of prime numbers from 1 to 30

These are {2, 3, 5, 7, 11, 13, 17, 19, 23, and 29}

Which are 10 in number

$$\text{Therefore the required probability} = \frac{\text{Number of favourable cases}}{\text{Total number of out comes}}$$

$$=\frac{10}{30} = \frac{1}{3}$$

Example 1.45. Out of 20 consecutive integers two are drawn at random. Then what is the probability that their sum is odd?

Solution

The total number of out comes of choosing 2

$$\text{Out of 20 consecutive integers are } 20_{C_2} = \frac{\angle 20}{\angle 2\ \angle 20-2}$$

$$= \frac{20 \times 19 \times \angle 18}{2\ \ \angle 18} = 190$$

Out of these 20 numbers even numbers are

= {2, 4, 6, 8, 10, 12, 14, 16, 18, and 20}

= {Total in number 10}

Out of these 20 numbers odd numbers are

= {1, 3, 5, 7, 9, 11, 13, 15, 17, and 19}

= {total in number 10}

For the sum of the chosen two numbers to be odd, one should be even and other should be odd.

The number of favorable cases to be one number is even = $10_{C_1} = 10$

Similarly the

Total numbe

Required pro

Example 1.46. If a
comes A, B, C, cl
permissible.

a. P (A) = 0.

b. P (A) = 0.

c. P (A) = 0.

Solution

a. P (A) + P (

The assign
each event

b. P (A) = 0.3

The assign
is negative.

c. P (A) = 0.3

P (A) + P (

The assignme
events is exceeds 1.

Example 1.47. Wha
three coins are tossed

Solution

In the tossing c

S = {H

= {T

Let E be the ev

E = {HHT,

Similarly the number of favorable cases to be one number is odd

$$= 10_{C_1} = 10$$

Total number of favorable cases that their sum is odd = 10×10

$$\text{Required probability} = \frac{\text{Number of favourable cases}}{\text{Total number of out comes}}$$

$$= \frac{10 \times 10}{190} = \frac{10}{19}$$ **Ans.**

Example 1.46. If an experiment has the three possible and mutually exclusive out comes A, B, C, check in each case whether the assignment of probabilities is permissible.

a. P (A) = 0.59 P (B) = 0.22 P(C) = 0.19
b. P (A) = 0.35 P (B) = 0.65 P(C) = - 0.03
c. P (A) = 0.36 P (B) = 0.54 P(C) = 0.33

Solution

a. P (A) + P (B) + P(C) = 0.59 + 0.22 + 0.19
= 1.00

The assignment is permissible because the values of the probability of each event is between 0 to 1 and their sum is 1.

b. P (A) = 0.35 P (B) = 0.65 P(C) = - 0.03

The assignment is not permissible because the probability of the event C is negative.

c. P (A) = 0.36, P (B) = 0.54, P(C) = 0.33
P (A) + P (B) + P(C) = 0.36 + 0.54 + 0.33
= 1.23

The assignment is not permissible because the sum of the probabilities of all events is exceeds 1.

Example 1.47. What is the probability of obtaining two heads and one tail when three coins are tossed?

Solution

In the tossing of three coins sample space is

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
= {Total m numbers 8}

Let E be the event of obtaining two heads and one tail. Then

E = {HHT, HTH, THH}

= {Total in numbers 3}

$$P(E) = \frac{\text{Number of favourable cases}}{\text{Total number of out comes}}$$

$$= \frac{3}{8} \qquad \textbf{Ans.}$$

Example 1.48. From a book of 100 pages, numbered 1 to 100. What is the probability that to get the number of the page is a perfect square?

Solution

In a book of 100 pages numbered 1 to 100,

Sample space is S = {1, 2, 3,.........100}

= {Total in numbers 100}

Let E be the event of drawing a page which is the perfect square number

E = {1, 4, 9, 16, 25, 36, 49, 64, 81, 100}

= {Total in numbers 10}

$$\text{Required probability} = \frac{10}{100} = \frac{1}{10} \qquad \textbf{Ans.}$$

Example 1.49. For any three arbitrary events A, B, C. Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$ **(Reg. Nov. 2006)**

Proof :

By taking left hand side

$$\text{L.H.S} = P(A \cup B \cup C)$$

$$= P(A \cup (B \cup C))$$

$$= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \quad \text{[By additive theorem]}$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap (B \cup C))$$

[By additive theorem in P (B∪C)]

$$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$$

[By set theory]

$$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))]$$

[by additive theorem for (A∩B) ∪ (A∩C)]

$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

[Since (A∩B) ∪ (A∩C) = A∩ B ∩ C]

= R. H. S.

$\Rightarrow$ P (A$\cup$B$\cup$C) = P (A) + P (B) + P(C) – P (A$\cap$B) – P (B$\cap$C) – P(C$\cap$A) + P (A$\cap$B$\cap$C) Proved.

Example 1.50. Define a random experiment. Sample space, event and mutually exclusive events. Give example of each.

(Supple. Feb. 2007 Set 4)
(Reg. Apr. /May 2005 Set. 2)

Solution

Random Experiment: The experiment which can be repeated any number of times under identical conditions, for such experiment all possible out comes are known in advance but the actual out come in a particular case in not known in advance. Then the experiment is called random experiment.

Sample Space: As defined on the page no.1.1

Events: As defined on the page no.1.1

Mutually Exclusive Events: As defined on the page no.1.2

Example 1.51. Find the probability of getting black king from a pack of 52 cards.

Solution

Total number of out comes are 52

In a pack of 52 cards, no. of black kings are 2.

i.e. Total number of favorable cases = 2

Probability to get black king from a pack of 52 cards = $\dfrac{2}{52} = \dfrac{1}{26}$

Example 1.52. A industry has 10 male and 5 women workers. Three workers are selected at random, one after the other. Find probability that

a. First two workers are male and third is female.

b. First and third of same sex and the second is opposite sex.

c. First and third males and second is female.

Solution

M $\rightarrow$ Male

F $\rightarrow$ Female

a. First two workers are male and third is female

$$P(M_1M_2F_3) = P(M_1) \,.\, P\left(\frac{M_2}{M_1}\right) . P\left(\frac{F_3}{M_1M_2}\right)$$

$$=\frac{10}{15}\times\frac{9}{14}\times\frac{5}{13}$$

$$=\frac{15}{91} \qquad \textbf{Ans.}$$

b. First and third of same sex and the second is of opposite sex

$$P(M_2\,F_2\,M_3)+P(F_1\,M_2\,F_3)$$

$$=P(M_1).P\left(\frac{F_2}{M_1}\right).P\left(\frac{M_3}{M_1\,F_2}\right)+P(F_1)\,.\,P\left(\frac{M_2}{F_1}\right).P\left(\frac{F_3}{F_1M_2}\right)$$

$$=\left(\frac{10}{15}\right)\times\left(\frac{5}{14}\right)\times\left(\frac{9}{13}\right)+\left(\frac{5}{15}\right)\times\left(\frac{10}{14}\right)\times\left(\frac{4}{13}\right)$$

$$=\frac{5}{21} \qquad \textbf{Ans.}$$

c. First and third males and second is female

$$P(M_1\,F_2\,M_3)=P(M_1)P\left(\frac{F_2}{M_1}\right).P\left(\frac{M_3}{M_1\,F_2}\right)$$

$$=\left(\frac{10}{15}\right)\times\left(\frac{5}{14}\right)\cdot\left(\frac{9}{13}\right)$$

$$=\frac{15}{7\times 13}$$

$$=\frac{15}{91} \qquad \textbf{Ans.}$$

Example 1.53. What is the probability of choosing any 3 or any Diamond or **both**?

Solution: we know that

Method – I

The Cards which may be '3' are total 4 the cards which may be diamond are total 13 but one card is common in both, means there are 16 playing cards which are either a '3' or a Diamond or both. Therefore

P (3 or Diamond or both) = 16/52 **Ans.**

Method – II

P (A or B or both) = P(A) + P(B) – P(A∩B)

[not mutually exclusive events]

P (any 3 or any Diamond or both)

Example 1.54. Wh

Solution we k

Method-I

The number o

P (4 or 7) = –

5

Method-II

P(any 4 or any

= P

= –

5

= –

5

Example 1.55. Wha

Solution

In the 3 dice tc

Number of pos

Required prob

Similarly prob

Example 1.56. Wha

are tossed?

Solution

Total number c

4 coins = 2^4 =

$$= P(\text{any } 3)+P(\text{any diamond}) - P(3 \text{ of diamond})$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$

$$= \frac{16}{52}$$ **Ans.**

Example 1.54. What is the probability of choosing any 4 or 7?

Solution we know that

Method-I

The number of cards which are any '4' or any '7' is 8 therefore

$$P\,(4 \text{ or } 7) = \frac{8}{52}$$ **Ans.**

Method-II

P(any 4 or any 7)

$$= P\,(\text{any } 4) + P\,(\text{any } 7) - P\,(\text{of the common event})$$

$$= \frac{4}{52} + \frac{4}{52} - \frac{0}{52}$$

$$= \frac{8}{52}$$ **Ans.**

Example 1.55. What is the probability of obtaining 9, 10, and 11 points with 3 dice?

Solution

In the 3 dice total number of combination will be 216.

Number of possible cases for obtaining 9 = 25

$$\text{Required probability} = \frac{25}{216}$$

Similarly probability for getting 10 and 11 are respectively $\frac{27}{216}$ and $\frac{27}{216}$

Example 1.56. What is the probability of getting 2 tails and 2 heads when 4 coins are tossed?

Solution

Total number of possible cases with

4 coins = 2^4 = 16

In the experiment of tossing of 4 coins sample space is

S = {HHHH, HTHH, T HHH, HHTH, HHHT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, TTHH, THTH, THHT, TTTT}

Total number of favorable cases of getting two heads and two tails are (HHTT, HTHT, HTTH, THHT, TTHH, THTH)

Total number 6

$$\text{Required probability} = \frac{6}{16} = \frac{3}{8} \qquad \textbf{Ans.}$$

Example 1.57. Let X denotes the number of heads in a single toss of 4 fair coins. Determine

1. $P(X < 2)$

2. $P(1 < X \le 3)$ **(Supple. Nov.2008)**

Solution

In the experiment of tossing of 4 coins sample space is

S = {HHHH, HTHH, T HHH, HHTH, HHHT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, TTHH, THTH, THHT, TTTT}

Total numbers of possible cases are 16.

Number of heads may be 0 then probability is = 1/16

Number of heads may be 1 then probability is = 4/16

Number of heads may be 2 then probability is = 6/16

Number of heads may be 3 then probability is = 4/16

Number of heads may be 4 then probability is = 1/16

X is the random variable of number of number of heads.

1. $P(X < 2) = P(X = 0) + P(X = 1)$

$$= \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

2. $P(1 < X \le 3) = P(X = 2) + P(X = 3)$

$$= \frac{6}{16} + \frac{4}{16}$$

$$= \frac{10}{16}$$

$$= \frac{5}{8}$$

Example 1.58. A c
there are at least tw
$P(E_1/E_2)$.

Solution

In the experim

In succession t

Is S= {HHH, H

No.	(
1	
2	
3	
4	
5	
6	
7	
8	

Sample space S_1 of tl

Sample space S_2 of tl

$P(E_1 \cap E_2) = P(I$

$P(E_1/E_2) = \frac{P(E}{I}$

$P(E_2) = 4 / 8 =$

Example 1.58. A coin is tossed three times in succession. If E_1 is the event that there are at least two heads and E_2 is the event that first throw gives a head, find $P(E_1/E_2)$.

Solution

In the experiment of tossing of a coin

In succession three times sample space

Is S= {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

No.	Coin 1st	Coin 2nd	Coin 3rd	Probabilities
1	H	H	H	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
2	H	H	H	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
3	H	T	H	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
4	T	H	T	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
5	H	T	T	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
6	T	T	H	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
7	T	H	T	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
8	T	T	T	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

Sample space S_1 of the event E_1 is

$$S_1 = \{ HHH, HHT, HTH, THH\}$$

Sample space S_2 of the event E_2 is

$$S_2 \{ HTT, THT, TTH, TTT\}$$

$$P(E_1 \cap E_2) = P(E_1/E_2)\, P(E_2)$$

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$P(E_2) = 4 / 8 = 1 / 2$$

$P(E_1 \cap E_2) = 3/8$ (In S_1 and S_2 common events are 3)

$$P(E_1/E_2) = \frac{3/8}{1/2}$$

$$= \frac{3}{4}$$ **Ans.**

Example 1.59. Given two independent events A, B such that P(A) = 0.20 and P(B) = 0.70

Determine
1) P (A and B)
2) P (A and not B)
3) P (not A and B)
4) P (neither A nor B)
5) P (A or B)

Solution

1) P (A and B) $= P(A). P(B)$
$= (0.20) \times (0.70)$
$= 0.14$

2) P (A and not B) $= P(A). P(B^C)$
$= (0.20) \times (1-0.70)$
$= (0.20)(0.30)$
$= 0.06$

3) P (not A and B) $= P(A^C). P(B)$
$= (1-0.20) \times (0.70)$
$= (0.80)(0.70)$
$= 0.56$

4) P (neither A nor B) $= P(A^C). P(B^C)$
$= (1-0.20)(1-0.70)$
$= (0.80)(0.30)$
$= 0.24$

5) P (A or B) $= 1 - P$ (neither A nor B)
$= 1 - 0.24$
$= 0.76$

Example 1.60.
probability of R
Find
1. Onl
2. Both
3. Non

Solution

1) Only one of t

2) Both of th

3) None of th

Example 1.61.
chance that the s

Solution

A page car
of getting a page
{9, 18, 27,

Required p

s.

P(A) = 0.20 and

Example 1.60. Ram and Shyam appear in an interview for two vacancies. The probability of Ram's selection is 1/5 and the probability of Shyam's selection is 1/7. Find

1. Only one of them will be selected
2. Both of them will be selected
3. None of them will be selected

Solution

1) Only one of them will be selected

$$= P(R).P(\bar{S}) + P(\bar{R})\ P(S)$$

$$= \left(\frac{1}{5}\right)\left(1-\frac{1}{7}\right)+\left(1-\frac{1}{5}\right)\left(\frac{1}{7}\right)$$

$$= \left(\frac{1}{5}\right)\left(\frac{6}{7}\right)+\left(\frac{4}{5}\right)\left(\frac{1}{7}\right)$$

$$= \frac{6}{35}+\frac{4}{35}=\frac{10}{35}=0.285$$ **Ans.**

2) Both of them will be selected

$$= P(R)\ P(S) = \left(\frac{1}{5}\right)\left(\frac{1}{7}\right)=\frac{1}{35}=0.0285$$ **Ans.**

3) None of them will be selected

$$= P(\bar{R})\times P(\bar{S})$$

$$= \left(1-\frac{1}{5}\right)\times\left(1-\frac{1}{7}\right)$$

$$= \frac{4}{5}\times\frac{6}{7}=\frac{24}{35}=0.685$$ **Ans.**

Example 1.61. A book contains 100 pages. A page chosen at random. What is the chance that the sum of digits on a page is equal to 9?

Solution

A page can be chosen from 100 pages in 100 ways number of favorable ways of getting a page having the sum of digits on it 9 is = 10

{9, 18, 27, 36, 45, 54, 63, 72, 81, 90}

Required probability is $= \frac{10}{100}=\frac{1}{10}=0.1$ **Ans.**

Example 1.62. A university has to select a mathematics teacher from a list of 50 persons. 20 of them are men and 30 women. 10 of them knowing Hindi and 40 not. 15 of them from Hyderabad and the remaining 35 from out of City. What is the probability of the university selecting a teacher who is Hindi knowing women from Hyderabad.

Solution

$$\text{Probability of selecting a women} = \frac{30}{50},$$

$$\text{Probability of selecting a teacher from Hyderabad} = \frac{15}{50}$$

$$\text{Probability of selecting a Hindi – knowing teacher} = \frac{10}{50}$$

The probability of the university selecting teacher

Who is Hindi knowing women from Hyderabad

$$= \frac{30}{50} \times \frac{10}{50} \times \frac{15}{50}$$

$$= \frac{9}{250} \qquad \textbf{Ans.}$$

Example 1.63. Ram writes three letters for his three friends and addresses three envelopes. These letters are placed at random and dispatched. What is the probability that no friend receives the correct letters?

Solution

Suppose the letters no. are 1, 2, 3

And envelope number are also 1, 2, 3

For keeping letter number 1, 2, 3 in the envelopes number 1, 2, 3. We have total

6 ways	1	2	3
	2	3	1
	3	1	2
	1	3	2
	2	1	3
	3	2	1

or we can say

the total number of ways of placing 3 letters in 3 envelopes = $|\underline{3}$

All the letters can be placed correctly in one way

i.e., 1, 2, 3

Hence its probability = 1/6

Required probability that all letters are sent

$$\text{Wrongly} = 1-\frac{1}{6}=\frac{5}{6} \qquad \textbf{Ans.}$$

Example 1.64. In a quiz programme, the probability that A will solve a problem is 3/4 and the probability that B will solve that is 1/4, what is the probability that the problem is at all solved?

Solution Given that

$$P(A)=\frac{3}{4}$$

$$P(B)=\frac{1}{4}$$

The probability that the problem is solved

$$= P(A \cup B)$$

$$= 1 - P(A \cup B)^C$$

$$= 1 - [P(A^C) \cap P(B^C)] \qquad \text{(by De-Morgan's law)}$$

$$= 1 - \left(1-\frac{3}{4}\right)\left(1-\frac{1}{4}\right) \qquad \text{(Both events are independent)}$$

$$= 1 - \frac{1}{4} \times \frac{3}{4}$$

$$= 1 - \frac{3}{16} \qquad = \frac{13}{16} \qquad \textbf{Ans.}$$

Example 1.65. A die is so loaded that the probability of each number appearing is proportional to the twice of the number. What is the probability of all events? What is the probability of occurrence of an odd number and greater or equal to five

Solution

Given that

Probability of each number appearing is proportional to the twice of the number

$$P(1) \propto 2 \times 1 \Rightarrow P(1) = 2k$$

$$P(1) \propto 2 \times 2 \Rightarrow P(2) = 4k$$

$$P(1) \propto 2 \times 3 \Rightarrow P(3) = 6k$$

$$P(1) \propto 2 \times 4 \Rightarrow P(4) = 8k$$

$$P(1) \propto 2 \times 5 \Rightarrow P(5) = 10k$$

$$P(1) \propto 2\times 6 \Rightarrow P(6) = 12k$$

We know that

$$P(1)+P(2)+P(3)+P(4)+P(5)+P(6)=1$$

$$2k+4k+6k+8k+10k+12k=1$$

$$42k=1$$

$$k=\frac{1}{42}$$

$$\Rightarrow \quad P(1)=2k=\frac{2}{42}=\frac{1}{21}$$

$$P(2)=4k=\frac{4}{42}=\frac{2}{21}$$

$$P(3)=6k=\frac{6}{42}=\frac{1}{7}$$

$$P(4)=8k=\frac{8}{42}=\frac{4}{21}$$

$$P(5)=10k=\frac{10}{42}=\frac{5}{21}$$

$$P(6)=12k=\frac{12}{42}=\frac{6}{21}$$

1) The probability of occurrence of an odd number

Odd number may be 1 or 3 or 5

$$\text{P(1) or P(3) or P(5)} = \frac{1}{21}+\frac{1}{7}+\frac{5}{21}$$

$$=\frac{9}{21}$$ **Ans.**

2) The probability that the number is greater or equal to five

That number may be 5 or 6

$$P(5)\ or\ P(6)=\frac{5}{21}+\frac{6}{21}$$

$$=\frac{11}{21}$$ **Ans.**

Example 1.66.
least one head?

Solution

In a fair c

The prob

The prob

The prob

Thus, the

One head

Example 1.67.
students speak E
students who spe

Solution

60% of the

70% of the

65% of the

The probab

$P(A^C \cup B^C)$

Thus the pe

Example 1.68. A
H&S department,
members are to be

1. Three
2. Two m

Example 1.66. A fair coin is tossed 6 times. What is the probability of getting at least one head?

Solution

In a fair coin

The probability of getting a head = 1/2

The probability of getting a tail = 1/2

The probability of getting no head in 6 tosses

$$P(E) = (1/2)^6 = 1/64$$

Thus, the probability of getting at least

$$\begin{aligned} \text{One head} &= 1 - \text{Probability of getting no head} \\ &= 1 - (1/2)^6 \\ &= 1 - 1/64 \\ &= \frac{64-1}{64} \\ &= \frac{63}{64} \end{aligned}$$

Example 1.67. In a classroom 60% of the students speak Hindi 70% of the students speak English, and 65% both Hindi and English. Find the percentage of the students who speak neither Hindi nor English

Solution

60% of the students speak Hindi $P(A) = 0.60$

70% of the students speak English $P(B) = 0.70$

65% of the students speak both Hindi and English $P(A \cap B) = 0.65$

The probability of the students who speak neither Hindi nor English

$$\therefore P(A^C \cup B^C) = ?$$

$$\begin{aligned} P(A^C \cup B^C) &= P(A \cap B)^C \text{ by D – Morgan's law} \\ &= 1 - P(A \cap B) \\ &= 1 - 0.65 \\ &= 0.35 \end{aligned}$$

Thus the percentage of the students who speak neither Hindi nor English is 35%.

Example 1.68. A culture committee consists of 9 members two of which are from H&S department, 3 form E.C.E. department and 4 from EEE department. Three members are to be removed at random. What is the probability that

1. Three members are from different departments
2. Two members from the same department and third from different department

ve

3. All three members are from the same department

Solution

Total number of ways of choosing 3 members out of 9 is

$$9c_3 = \frac{9!}{3!\ \ 9-3!} = \frac{9\times8\times7}{3\times2\times1} = 84$$

1. Three removed members are from different department one member from H & S department out of two $= 2_{c_1}$

$= 2$

One member from E.C.E department out of three $= 3_{c_1}$

$= 3$

One member from E.E.E department out of four $= 4_{c_1}$

$= 4$

Three members from different departments

i.e. required probability $= \dfrac{2\times3\times4}{84} = \dfrac{2}{7}$

2. Two members from same department and third from different department

The number of ways of removing two members from H & S department and one from others $= 2_{c_2} \times 7_{c_1}$

The number of ways of removing two member from ECE department and one from others $= 3_{c_2} \times 6_{c_1}$

The number of ways of removing two members from EEE department and one from others $= 4_{c_2} \times 5_{c_1}$

Two members from same department and third from different department

i.e. required probability $= \dfrac{2_{c_2}\times7_{c_1} + 3_{c_2}\times6_{c_1} + 4_{c_2}\times5_{c_1}}{84}$

$$= \frac{55}{84}$$

3. All three number one from same department the number of ways of removing three membe. from ECE department $= 3_{c_3} = 1$ way

The number of ways of removing three numbers from EEE department

$$= 4_{c_3} = 4 \text{ ways}$$

All three members one from same department i.e required probability

$$= \frac{1+4}{84} = \frac{5}{84}$$

Example 1.69. A pair of dice tossed twice. Find the probability of scoring 5 points

a) Once b) Twice c) At least once

Solution

In the experiment of tossing of a pair of dice, number of possible out comes in which we get 5 are (1, 4) (2, 3) (3, 2) (4, 1) i.e in 4 ways, then

The probability of getting $5 = \frac{4}{36} = \frac{1}{9}$

The probability of not getting $5 = 1 - \frac{1}{9} = \frac{8}{9}$

a. If a pair of dice tossed twice then the probability of scoring 5 points in one toss, one toss may be first or second

The probability of getting 5 in the first toss and not getting 5 in the second toss

$$= \frac{1}{9} \times \frac{8}{9} = \frac{8}{81}$$

Similarly, the probability of not getting 5 in the first toss and getting 5 in the second toss

$$= \frac{8}{9} \times \frac{1}{9} = \frac{8}{81}$$

Required probability $= \frac{8}{81} + \frac{8}{81} = \frac{16}{81}$

b. The probability of getting 5 in both tosses $= \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$

c. The probability of getting 5 in at least one toss

At least one toss means may be first as may be second or may be both

Required probability $= \frac{1}{9} \times \frac{8}{9} + \frac{8}{9} \times \frac{1}{9} + \frac{1}{9} \times \frac{1}{9}$

$$= \frac{8}{81} + \frac{8}{81} + \frac{1}{81}$$

$$= \frac{17}{81}$$

Example 1.70. Ram can hit a target 2 times in 4 shots, Shyam 3 times in 5 shots and Hari 3 times in 4 shots. What is the probability that

1. Two shots hit
2. At least two shots hit

Solution

Probability of Ram hitting the target $= 2/4$

Probability of Shyam hitting the target $= 3/5$

Probability of Hari hitting the target $= 3/4$

1. Two shots hit

Two shots hit means may be Ram and Shyam or Shyam and Hari or Ram, Hari

Probability that Ram and Shyam hit the target and Hari fails

$$= \frac{2}{4} \times \frac{3}{5} \left(1 - \frac{3}{4}\right)$$

$$= \frac{2}{4} \times \frac{3}{5} \times \frac{1}{4} = \frac{3}{10}$$

Probability that Shyam and Hari hits the target and Ram fails

$$= \frac{3}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{4}\right)$$

$$= \frac{3}{5} \times \frac{3}{4} \times \frac{2}{4} = \frac{9}{40}$$

Probability that Ram and Hari hits the target and Shyam fails

$$= \frac{2}{4} \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right)$$

$$= \frac{2}{4} \times \frac{3}{4} \times \frac{2}{5} = \frac{6}{40} = \frac{3}{20}$$

2. At least two shots hit at least two shorts means may be two or may be all three shots also

Probability that all three shots hit $= \frac{2}{4} \times \frac{3}{5} \times \frac{3}{4} = \frac{9}{40}$

Required probability $= \frac{27}{40} + \frac{9}{40} = \frac{36}{40} = \frac{9}{10}$

Example 1.71. A problem of mathematics is given to three students A, B and C whose chances of solving the problems are 1/3, 1/4 and 1/5 respectively.

What is the probability that the problem will be solved?

Solution

Method – I

The probability that A can solve the problem is 1/3

The probability that A can not solve the problem is (1 – 1/3)

The probability that B can solve the problem is 1/4

The probability that B can solve the problem is (1 – 1/4)

The probability that C can not solve the problem is 1/5

The probability that B can not solve the problem is (1 – 1/5)

Required probability that the problem will be solved

$$= 1-(1-1/3)\,(1-1/4)\,(1-1/5)$$

$$= 1-(2/3)\times(3/4)\times(4/5)$$

$$= 1-24/60$$

$$= \frac{36}{60}$$

$$= 3/5$$

Method – II

At least one student can solve the problem, at least one means may be two students or may be all three can solve the problem probability that A solve the problem and B and C can not $= \dfrac{1}{3}\left(1-\dfrac{1}{4}\right)\left(1-\dfrac{1}{5}\right)$

Probability that B solves the problem and C and A can not

$$= \frac{1}{4}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

Probability that C solve the problem and A and B can not

$$= \frac{1}{5}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)$$

Probability that any one of them solve the problem

$$= \frac{1}{3}\left(1-\frac{1}{4}\right)\left(10\frac{1}{5}\right)+\frac{1}{4}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)+\frac{1}{5}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{3}\times\frac{3}{4}\times\frac{4}{5}+\frac{1}{4}+\frac{1}{4}\times\frac{2}{3}\times\frac{4}{5}+\frac{1}{5}\times\frac{2}{3}\times\frac{3}{4}$$

$$= \frac{12}{60}+\frac{8}{60}+\frac{6}{60}=\frac{26}{60}$$

Similarly probability that any two of them solve the problem

$$= \frac{1}{3} \times \frac{1}{4}\left(1-\frac{1}{5}\right) + \frac{1}{4} \times \frac{1}{5}\left(1-\frac{1}{3}\right) + \frac{1}{3} \times \frac{1}{5}\left(1-\frac{1}{4}\right)$$

$$= \frac{40}{60} + \frac{2}{60} + \frac{3}{60}$$

$$= \frac{9}{60}$$

Probability that all three solve the problem

$$= \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}$$

$$= \frac{1}{60}$$

Required probability that at least one of them solve the problem or probability that problem will be solved

$$= \frac{26}{60} + \frac{9}{60} + \frac{1}{60}$$

$$= \frac{36}{60} = \frac{3}{5}$$

Example 1.72. A patient is selected at random from 80 patients where 30 are suffering from high blood pressure, 20 from head ache and 10 from both high blood pressure & headache. Find the probability P that the patient is suffering from high blood pressure or headache

Solution

Given that B $\rightarrow$ Blood pressure, H $\rightarrow$ Head ache.

$$P(B) = \frac{30}{80} = \frac{3}{8}$$

$$P(H) = \frac{20}{80} = \frac{1}{4}$$

$$P(B \cap H) = \frac{10}{80} = \frac{1}{8}$$

Thus by the addition rule

$$P(B \cup H) = P(B \text{ or } H)$$

$$= P(B) + P(H) - P(B \cap H)$$

Example 1.73. I
are in the compet
(I), IT (I) will ge
getting high score

Solution

Suppose the

It will get h

CSE will ge

We know th

Example 1.74. A
inches. Find the pro

Solution

Suppose sam
A denote the set of

$$= \frac{3}{8} + \frac{1}{4} - \frac{1}{8}$$

$$= \frac{3 + 2 - 1}{8}$$

$$= \frac{4}{8} = \frac{1}{2}$$

Example 1.73. In an engineering college, three branches CSE (C), IT (I), ECE (E) are in the competition. The chance that CSE (C) will get high score is twice of IT (I), IT (I) will get high score is thrice of ECE (E). Find out the probabilities of getting high score, for the mentioned three branches, P(C), P (I), P (E).

Solution

Suppose the chance that branch ECE will get high score is P

$$P\ (ECE) = P\ (E) = P$$

It will get high score, it is the thrice of ECE

$$P\ (I) = 3P$$

CSE will get high score, it is the twice of it

$$P\ (C) = 2 \times 3P = 6P$$

We know that the sum of the probabilities must be 1 hence

$$P\ (C) + P\ (I) + P\ (E) = 1$$

$$6P + 3P + P = 1$$

$$10\ P = 1$$

$$P = 1/10$$

$$P\ (E) = P = 1/10$$

$$P\ (I) = 3P = 3 \times \frac{1}{10} = \frac{3}{10}$$

$$P\ (C) = 2 \times 3P = \frac{6 \times 1}{10} = \frac{6}{10}$$

Example 1.74. A point is selected at random inside a rectangle measuring 3 by 4 inches. Find the probability that the point is at least 1 inch from the edge

Solution

Suppose sample space S denote the set of points inside the rectangle and let A denote the set of points at least 1 inch from the edge

$$P = \frac{\text{area of A}}{\text{area of S}} = \frac{1 \times 2}{3 \times 4} = \frac{2}{12} = \frac{1}{6}$$

Example 1.75. There are 20 persons in a birthday party. What is the probability that at least two of them have the same birthday?

Solution

The probability that n people have distinct birthdays where $n \leq 365$

There are n people and 365 different days. There one 365^n ways in which the n people can have their birthday.

The first person can be born on any of the 365 days

The second person can be born on the remaining 364 days

Similarly and so on

Thus the number of ways that n persons can have distinct birthdays

$$365 \times 364 \times 363 \ldots\ldots\ldots\ldots \times [365 - (n - 1)]$$

$$365 \times 364 \times 363 \ldots\ldots\ldots\ldots \times [365 - (n + 1)]$$

There fore

$$\text{P (n people have distinct birthdays} = \frac{365 \times 364 \times \ldots\ldots (365 - n + 1)}{(365)^n}$$

Thus the probability P that two or more people have same birthday

P (at least two out of n people have the same birthday)

$$= 1 - \text{P (n people have distinct birthdays)}$$

$$= 1 - \frac{365 \times 364 \times \ldots\ldots (365 - n + 1)}{(365)^n}$$

In the question we have to find at beast two people out of 20 persons have same birthday n = 20

P (at least two out of 20 people have the same birthday)

$$= 1 - \frac{365 \times 364 \times \ldots\ldots\ldots (365 - 20 + 1)}{(365)^{20}}$$

$$= 1 - \frac{365 \times 364 \times \ldots\ldots\ldots \times 346}{(365)^{20}}$$

$$= 1 - \frac{1.03669 \times 10^{51}}{1.761397 \times 10^{51}}$$

$$= 1 - 0.58856$$

$$= 0.41143$$

Example 1.76. T
5/10, 3/10, 2/10.
become M.D.'s ar

Solution

The Probabi
The Probabi
The Probabi
Bonus schen
Bonus schen
Bonus schen
Required Pr
P(BS) = P(A

$= \frac{5}{10} \times 0.02$

= 0.01 + 0.(

= 0.031

$P(A/BS) =$

$P(B/BS) =$

$P(A/BS) =$

Example 1.76. The Probabilities of A, B, C to become M.D.'s of a factory are 5/10, 3/10, 2/10.The Probabilities that bonus scheme will be introduced if they become M.D.'s are 0.02, 0.03, and 0.04. Find bonus scheme introduced.

(Supple Nov.2008)

Solution

The Probabilities of A to become M.D.'s of a factory is P(A) = 5/10.

The Probabilities of B to become M.D.'s of a factory is P(B) = 3/10

The Probabilities of C to become M.D.'s of a factory is P(C) = 2/10.

Bonus scheme will be introduced if A become M.D. P(BS/A) = 0.02

Bonus scheme will be introduced if B become M.D. P(BS/B) = 0.03

Bonus scheme will be introduced if C become M.D. P(BS/C) = 0.04

Required Probability

P(BS) = P(A) P(BS/A) + P(B) P(BS/B) + P(C) P(BS/C)

$$= \frac{5}{10} \times 0.02 + \frac{3}{10} \times 0.03 + \frac{3}{10} \times 0.04$$

$$= 0.01 + 0.009 + 0.012$$

$$= 0.031$$

$$P(A/BS) = \frac{P(A)P(BS/A)}{P(BS)} = \frac{\frac{5}{10} \times 0.02}{\frac{5}{10} \times 0.02 + \frac{3}{10} \times 0.03 + \frac{3}{10} \times 0.04}$$

$$= \frac{0.01}{0.031} = 0.322$$

$$P(B/BS) = \frac{P(B)P(BS/B)}{P(BS)} = \frac{\frac{3}{10} \times 0.03}{\frac{5}{10} \times 0.02 + \frac{3}{10} \times 0.03 + \frac{3}{10} \times 0.04}$$

$$= \frac{0.009}{0.031} = 0.290$$

$$P(A/BS) = \frac{P(C)P(BS/C)}{P(BS)} = \frac{\frac{3}{10} \times 0.04}{\frac{5}{10} \times 0.02 + \frac{3}{10} \times 0.03 + \frac{3}{10} \times 0.04}$$

$$= \frac{0.012}{0.031} = 0.387$$

Example 1.77. Suppose A = [P_1, P_2, P_3, P_4] and suppose P is a probability function defined on A, then

a. Find P (p_1) if p (p_2) = 0.3, p (p_3) = 0.4, p (p_4) = 0.1
b. Find p (p_1) and p(p_2) if p (p_3) = p (p_4) = 0.3 and p (p_1) = 2 p (p_2)

Solution We know that the sum of the

(a) Probabilities on the sample points must equal to one.

Suppose

$p(p_1) + p(p_2) + p(p_3) + p(p_4) = 1$

$p(p_1) + 0.3 + 0.4 + 0.1 = 1$

$p(p_1) = 1 - 0.8 = 0.2$

(b) $p(p_1) + p(p_2) + p(p_3) + p(p_4) = 1$

$2p(p_2) + p(p_2) + 0.3 + 0.3 = 1$

$3p(p_2) + 0.6 = 1$

$3p(p_2) = 1 - 0.6$

$3p(p_2) = 0.4$

$P(p_2) = 0.4/3$

$P(p_2) = 0.133$

We have $p(p_1) = 2p(p_2)$

$P(p_1) = 2 \times 0.133$

$P(p_1) = 0.266$

Example 1.78. Determine the probability of each event

1. An even number appears in the toss of a fair die
2. 2 or more heads appear in the toss of 4 fair coins
3. Exactly one 5 appears in the toss of 2 fair dice

Solution

(1) In the experiment of tossing of a fair die,

Total numbers of outcomes are 6

Numbers of possible cases are 2, 4 & 6

Required probability = 3/6 = 1/2

(2) Two are more heads appear in the toss of 4 fair coins

Total numbers of outcomes are 16 these are

[HHHH, HTHH, THHH, HHTH, HHHT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTT, TTHH, THTH, THHT, TTTTT}

ɔability function

p (p_2)

Number of favorable cases of getting two more heads are 11

Required probability = 11 / 16

(3) Exactly one 5 appears in the tossing of 2 fair dice

In the tossing of 2 fair dice total number of out comes are 36

Number of favorable cases that exactly one 5 appear are ten (10)

{(1, 5), (2, 5) (3, 5) (4, 5) (6, 5) 5, 1) (5, 2) (5, 3) (5, 4) (5, 6)}

Required probability = 10 / 36

Example 1.79 If P(A) = a, P (B) = b, P(A∩B) = c, express the following in terms of a , b and c

1) $P(A^c \cap B^c)$

2) $P(A^c \cap B)$

3) $P(A^c \cap P(A \cup B)))$

4) $P[A \cup (A^C \cap B)]$ **(Supple. Feb.2010 Set 1)**

Solution

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= a + b - c$$

1) By De Morgan's law

$(A \cup B)^C = A^C \cap B^C$ Thus

$$P(A^C \cap B^C) = P(A \cup B)^C$$
$$= 1 - (A \cup B)$$
$$= 1 - (a + b - c)$$
$$= 1 - a - b + c$$

2) We know that

$$P(B) = P(A \cap B) + P(A^C \cap B)$$
$$b = c + P(A^C \cap B)$$
$$b - c = P(A^C \cap B)$$

3)

$$P[A^C \cap (A \cup B)] = P(A^C \cap A) \cup P(A^C \cap B)$$
$$= \phi \cup P(A^C \cap B)$$
$$= P(A^C \cap B)$$
$$= b - c$$

THTT, TTHT,

4)

$$P[A\cup(A^c\cap B)] = P(A\cup A^C)\cap P(A\cup B)$$
$$= \bigcup\cap P(A\cup B)$$
$$= P(A\cup B)$$
$$= a+b-c$$

Example 1.80. What is the probability of getting either 2 or 3 in the experiment of rolling a single die?

Solution P (A or B) = P (A) + P (B)

P (2 or 3) = P (2) + P (3)

$$= \frac{1}{6}+\frac{1}{6}$$

$$= \frac{2}{6} = 1/3$$

Example 1.81. For the discrete probability distribution

X	0	1	2	3	4	5	6	7
F	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

(Nov. 2004 Supple. Set 4)
(Nov. /Dec. 2005 Set 4)
(Nov.2009 set 1)

Determine

1. k
2. mean
3. variance
4. smallest value of x^* $s.t. P(x\le x^*)>\frac{1}{2}$

Solution

1. $\sum_{all\,x} f(x) = \sum_{all\,x} P(X=x) = 1$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$
$$10\,k^2 + 9k = 1$$

k

2. Mean $=\sum$

= 0.0

+

= k +

= 30k

=30

=3 +

= 15(

3. Variance σ

$=(1-3.66)$

$+(4-3.66)$

$=(7.0756)$

$+(5.4756)2$

k

= 0.70756 + 0.5

$\sigma^2 = 3.4044$

3. Smallest valu

experiment of

Supple. Set 4)
c. 2005 Set 4)
ov.2009 set 1)

$$10k^2 + 9k - 1 = 0$$

$$k = \frac{-9 \pm \sqrt{81 - 4 \times 10(-1)}}{2 \times 10} = \frac{-9 \pm \sqrt{81+40}}{20} = \frac{-9 \pm \sqrt{121}}{20}$$

$$= \frac{-9 \pm 11}{20} = \frac{-9+11}{20} \; or \; \frac{-9-11}{20}$$

$$= \frac{2}{20} \; or \; \frac{-20}{20}$$

$$= \frac{1}{10} \; or \; -1$$

Ans.

2. Mean $= \sum x \, f(x)$

$$= 0.0 + 1 \times k + 2 \times 2k + 3 \times 2k + 4 \times 3k + 5 \times k^2 + 6 \times 2k^2 + 7\,(7k^2 + k)$$

$$= k + 4k + 6k + 12k + 5k^2 + 12k^2 + 49k^2 + 7k$$

$$= 30k + 66k^2$$

$$= 30 \times \frac{1}{10} + 66 \times \frac{1}{100}$$

$$= 3 + \frac{33}{50}$$

$$= \frac{150+33}{50} = \frac{183}{50} = 3.66$$

3. Variance $\sigma^2 = \sum_i (x_i - \mu)^2 f(x_i)$

$$= (1-3.66)^2 k + (2-3.66)^2 2k + (3-3.66)^2 2k$$
$$+ (4-3.66)^2 3k + (5-3.66)^2 k^2 + (6-3.66)^2 2k^2 + (7-3.6)^2 (7k^2 + k)$$
$$= (7.0756)k + (2.7556) \times 2k + (0.4356) \times 2k + (0.1156)3k + (1.7956)k^2$$
$$+ (5.4756)2k^2 + (11.1556)(7k^2 + k)$$

$$k = \frac{1}{10}$$

$$= 0.70756 + 0.55112 + 0.08712 + 0.03468 + 0.017956 + 0.109512 + 1.896452$$

$\sigma^2 = 3.4044 \qquad \sigma = 1.845101$ **Ans.**

3. Smallest value of X^* $s.t. P(X \leq X^*) > \frac{1}{2}$

$$f(0) + f(1) = 0.1$$
$$f(0) + f(1) + f(2) = 0.3$$
$$f(0) + f(1) + f(2) + f(3) = 0.5$$
$$f(0) + f(1) + f(2) + f(3) + f(4) = 0.8$$

Smallest value of $X^* \ s.t. P(X \le X^*) > \frac{1}{2}$ is 4

$P(x \le 3) = 0.5, \quad P(x \le 4) = 0.8$

Example 1.82. If 3 cars are drawn from a lot of 6 cars containing 2 defective cars, find the probability distribution of the number of defective cars.

(Nov. 2004 Supple. Set 4)

(Nov. / Dec. 2005 Supple. Set 4)

Solution

Given that 3 cars are drawn from a lot of 6 cars,

Total number of possible cases $= 6_{C_3}$

$$= \frac{\angle 6}{\angle 3 \ \angle 3}$$

$$= \frac{6 \times 5 \times 4}{2 \times 2} = 20$$

Probability distribution of the number of detective cars =?

Number of defective cars may be 0, 1, or 2

If number of defective cars 0

Then total number of favorable cases = $2_{C_0} \ 4_{C_3}$

If number of defective cars 1

Then total number of favorable cases = $2_{C_1} \ 4_{C_2}$

If number of defective cars 2

The total number of favorable cases = $2_{C_2} \ 4_{C_1}$

Thus probability distribution of the number of detective cars

$X = x_i$	0	1	2
$P(X = x_i) = f(x)$	$\frac{2_{C_0} \ 4_{C_3}}{6_{C_3}}$	$\frac{2_{C_1} \ 4_{C_2}}{6_{C_3}}$	$\frac{2_{C_2} \ 4_{C_1}}{6_{C_3}}$

Example 1.83.
items of which 5

Solution

A sample c
which 5 are defe

Total numb

Defective it

Since we ha

1. If number o
Then total n

2. If number o
Then total n

3. If number o
Then total n

4. If number o
Then total n

5. If number of
The total nu

Thus the pro

$X = x_i$
$P(X = x_i) = P(x)$

Expectation =

=

=

=

Example 1.83. A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected number of defective items.

(Nov. 2004 Supple. Set 3)

Solution

A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective.

Total number of possible cases = 12_{C_4}

Defective item may be 0, 1, 2, 3, 4

Since we have to select 4 items.

1. If number of defective items are 0

   Then total number of favorable cases = $5_{C_0}\ 7_{C_4}$

2. If number of defective items are 1

   Then total number of favorable cases = $5_{C_1}\ 7_{C_3}$

3. If number of defective items are 2

   Then total number of favorable cases = $5_{C_2}\ 7_{C_2}$

4. If number of defective items are 3

   Then total number of favorable cases = $5_{C_3}\ 7_{C_1}$

5. If number of defective items are 4

   The total number of favorable cases = $5_{C_4}\ 7_{C_0}$

Thus the probability distribution of the defective items

X = x$_i$	**0**	**1**	**2**	**3**	**4**
P(X = x$_i$) = P(x)	$\dfrac{5_{C_0}\ 7_{C_4}}{12_{C_4}}$	$\dfrac{5_{C_1}\ 7_{C_3}}{12_{C_4}}$	$\dfrac{5_{C_2}\ 7_{C_2}}{12_{C_4}}$	$\dfrac{5_{C_3}\ 7_{C_1}}{12_{C_4}}$	$\dfrac{5_{C_4}\ 7_{C_0}}{12_{C_4}}$
	$\dfrac{1\times 35}{495}$	$\dfrac{5\times 35}{495}$	$\dfrac{10\times 21}{495}$	$\dfrac{10\times 7}{495}$	$\dfrac{5\times 1}{495}$

Expectation $= \sum x_i\, P(x_i)$

$$= \frac{0\times 35}{495} + \frac{1\times 5\times 35}{495} + \frac{2\times 10\times 21}{495} + \frac{3\times 10\times 7}{495} + \frac{4\times 5\times 1}{495}$$

$= 0.3535 + 0.8484 + 0.42424 + 0.04040$

$= 1.6666$ **Ans.**

Example 1.84. Let $f(x) = 3x^2$, when $0 \le x \le 1$ be the probability density function of a continuous variable X. Determine 'a' and 'b' such that

(Nov. 2004 Supple. Set 2)

1. $p(x \le a) = p(x > a)$
2. $p(x > b) = 0.5$

Solution

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx = 1$$

$$\int_{0}^{1} 3x^2\, dx = 1$$

$$3\left[\frac{x^3}{3}\right]_0^1 = 1$$

Given that

1. $p(x \le a) = p(x > a)$

$$\int_{-\infty}^{a} f(x)\, dx = \int_{a}^{\infty} f(x)dx$$

$$\int_{0}^{a} 3x^2\, dx = \int_{a}^{1} 3x^2\, dx$$

$$3\left[\frac{x^3}{3}\right]_0^a = 3\left[\frac{x^3}{3}\right]_a^1$$

$$\frac{a^3}{3} = \frac{1}{3} - \frac{a^3}{3}$$

$a^3 = 1 - a^3$

$2a^3 = 1$

$a^3 = \frac{1}{2}$

$a = (0.5)^{\frac{1}{3}}$

$a = 0.7937$ **Ans.**

2. $p(x > b) = 0.5$

$$\int_{b}^{\infty} f(x)dx = 0.5$$

$$\int_{b}^{1} f(x)dx = 0.5$$

$$3\left[\frac{x^3}{3}\right]_b^1 = 0.5$$

$$\left[x^3\right]_b^1 = 0.5$$

$1 - b^3 = 0.5$

$1 - 0.5 = b^3$

$(0.5)^{\frac{1}{3}} = b$

$b = 0.7937$ **Ans.**

Example 1.85.

$$f(x) = \begin{cases} \frac{1}{6}k \\ \end{cases}$$

Determine

1. the value of
2. the mean
3. $p(1 \le x \le 2)$

Solution

1. $\int_{-\infty}^{\infty} f(x$

Example 1.85. If x is a continuous random variable with distribution

(Nov 2006 Reg. Set 4)

$$f(x)=\begin{cases} \dfrac{1}{6}k+k & \text{if } 0\le x\le 3 \\ 0 & \text{else where} \end{cases}$$

Determine

1. the value of k
2. the mean
3. p(1 ≤ x ≤ 2)

Solution

1. $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_{-\infty}^{0} f(x)dx+\int_{0}^{3} f(x)\,dx+\int_{3}^{\infty} f(x)\,dx=1$$

$$\int_{-\infty}^{0} 0\,dx+\int_{0}^{3}\left(\frac{1}{6}x+k\right)dx+\int_{3}^{\infty} 0\,dx=1$$

$$\int_{0}^{3}\left(\frac{1}{6}k+k\right)dx=1$$

$$\left[\frac{1}{6}\frac{x^2}{2}+kx\right]_0^3=1$$

$$\frac{1}{12}\times\frac{9}{1}+3k=1$$

$$\frac{3}{4}+3k=1$$

$$3k=1-\frac{3}{4}$$

$$3k=\frac{1}{4}$$

$$k=\frac{1}{12}$$

Ans.

2. Mean = μ $= \int_{-\infty}^{\infty} f(x)dx$

$$= \int_{-\infty}^{0} x\, f(x)dx + \int_{0}^{3} x\, f(x)dx + \int_{3}^{\infty} x\, f(x)$$

$$= \int_{-\infty}^{0} k.0\, dx + \int_{0}^{3} x.\left(\frac{1}{6}x + k\right) dx + \int_{3}^{\infty} x.0\, dx$$

$$= \int_{0}^{3}\left(\frac{1}{6}x^2 + kx\right)dx$$

$$= \left[\frac{1}{6}\frac{x^3}{3} + k\frac{x^2}{2}\right]_0^3$$

$$= \frac{1}{18}\times 27 + \frac{k}{2}\times 9$$

$$= \frac{3}{2} + \frac{9}{2\times 12}$$

$$= \frac{3}{2} + \frac{3}{8} = \frac{12+3}{8} = \frac{15}{8}$$ **Ans**

3. $P(1 \le x \le 2) = \int_{1}^{2} f(x)dx$

$$= \int_{1}^{2}\left(\frac{1}{6}x + k\right) dx$$

$$= \left[\frac{1}{6}\frac{x^2}{2} + kx\right]_1^2$$

$$= \frac{1}{6}\,\frac{4}{2} + 2k - \frac{1}{12} - k$$

$$= \frac{1}{3} + 2\times\frac{1}{12} - \frac{1}{12} - \frac{1}{12}$$

$$= \frac{1}{3}$$ **Ans.**

Example 1.86.

F(x) = 0
= k (
= 1
Determine
1. f(x)
2. k
3. mean

Solution

We know F

$f(x) =$

$\int_{-\infty}^{\infty} f(x)d$

$\int_{-\infty}^{\infty} f(x)d$

$\int_{-\infty}^{1} 0\, dx +$

$4k\int_{1}^{3}(x-1$

$4k\left[\frac{(x-1)}{4}\right.$

$4k\times\frac{16}{4} = 1$

16k = 1

$k = \frac{1}{16}$

Mean $\mu = \int_{-\infty}^{\infty}$

$= \int_{-\infty}^{1}$

$= \int_{-\infty}^{1}$

Example 1.86. If F(x) is the distribution function of x given by

(Feb. Supple. 2007 Set 1)
(Nov. 2006 Reg. Set 3)
(Nov. /Dec. Supple. 2005 Set 3)

$$F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ k(x-1)^4 & \text{if } 1 < x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$

Determine

1. f(x)
2. k
3. mean

Solution

We know $F'(x) = f(x)$

$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 4k(x-1)^3 & \text{if } 1 < x \le 3 \\ 0 & \text{If } x > 3 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{1} 0\,dx + \int_{1}^{3} 4k(x-1)^3\,dx + \int_{3}^{\infty} 0\,dx = 1$$

$$4k\int_{1}^{3} (x-1)^3 dx = 1$$

$$4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$4k \times \frac{16}{4} = 1$$

$$16k = 1$$

$$k = \frac{1}{16}$$ **Ans.**

$$\text{Mean } \mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \int_{-\infty}^{1} x\,f(x)dx + \int_{1}^{3} x\,f(x)dx + \int_{3}^{\infty} x\,f(x)dx$$

$$= \int_{-\infty}^{1} x.0\,dx + \int_{1}^{3} x.4k(x-1)^3\,dx + \int_{3}^{\infty} x.0\,dx$$

$$= 4k \int_1^3 x\,(x-1)^3\,dx$$

$$= 4k\left[x \int (x-1)^3\,dx - \int \frac{d}{dx}x \int (x-1)^3\,dx\right] dx$$

$$= 4k\left(x.\frac{(x-1)^4}{4} - \int \frac{(x-1)^4}{4}\,dx = 4k\left[x.\frac{(x-1)^4}{4} - \frac{(x-1)^5}{20}\right]_1^3\right)$$

$$= 4k \times \left(12 - \frac{8}{5}\right) = \frac{1}{4} \times 52 = 13 \qquad \textbf{Ans.}$$

Example 1.87. If $f(x) = ke^{-|x|}$ is a probability density function in $-\infty < x < \infty$, find

(Feb. 2007 Supple. Set 3)

1. The value of k
2. The variance
3. The probability between 0 and 4

Solution

1. The value of k

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} ke^{-|x|}\,dx = 1$$

$$\int_{-\infty}^{0} ke^{-|x|}dx + \int_0^{\infty} ke^{-|x|}dx = 1$$

[**NOTE:** Since in $-\infty$ to 0, $|x| = -x$ and 0 to ∞, $|x| = x$]

$$\int_{-\infty}^{0} ke^{x}\,dx + \int_0^{\infty} ke^{-x}\,dx = 1$$

$$k\left[e^{x}\right]_{-\infty}^{0} + k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$k\left[e^{0} - e^{-\infty}\right] + k\left[-e^{-\infty} + e^{0}\right] = 1$$

$$K + k = 1$$

$$2k = 1$$

$$k = \frac{1}{2} \qquad \textbf{Ans.}$$

2. Variance

$$\text{Mean} = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{0} x\, f(x) dx + \int_{0}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{0} x\ \ ke^{-|x|} dx + \int_{0}^{\infty} x\ \ ke^{-|x|} dx$$

$$= k\int_{-\infty}^{0} xe^{x} dx + k\int_{0}^{\infty} xe^{-x} dx$$

$$= k\left[x\int e^{x} - \int\left(\frac{d}{dx}x\int e^{x}\, dx\right)dx\right]$$

$$- k\left[x.\int e^{-x} dx - \int\left(\frac{d}{dx}x\int e^{-x}\, dx\right)dx\right]$$

$$= k\left[xe^{x} - e^{x}\right]_{-\infty}^{0} \quad - k\left|-xe^{-x} - \int -e^{-x}\, dx\right|$$

$$= k\left[x\ e^{x} - e^{x}\right]_{-\infty}^{0} - k\left[-xe^{-x} - e^{-x}\right]_{0}^{\infty}$$

$$= k\left[e^{0} - e^{0} - (-\infty)e^{-\infty} + e^{-\infty}\right] + k\left[\infty e^{-\infty} - e^{-\infty} - 0e^{-0} + e^{-0}\right]$$

$$= k[-1] + k[1] \ = 0$$

Variance σ^2

$$E\left(x^2\right) = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|} dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} x^2 e^{x} dx + \frac{1}{2}\int_{0}^{\infty} x^2 e^{-x}\, dx$$

$$= \frac{1}{2}\left[x^2\int e^{x} dx - \int \frac{d}{dx}x^2 \int e^{x} dx\right]_{-\infty}^{0}$$

$$+ \frac{1}{2}\left[x^2\int e^{-x} dx - \int\left(\frac{d}{dx}x^2\int e^{-x}\right)dx\right]_{0}^{\infty}$$

$$= \frac{1}{2}\left[x^2 e^{x} - \int 2xe^{x}\right]_{-\infty}^{0} + \frac{1}{2}\left[-x^2 e^{-x} + \int 2xe^{-x} dx\right]_{0}^{\infty}$$

$$= \frac{1}{2}\left[x^2 e^{x} - 2(-1)\right]_{-\infty}^{0} + \frac{1}{2}\left[-x^2 e^{-x} + 2(1)\right]_{0}^{\infty}$$

$$= \frac{1}{2}[0 - \infty + 2] + \frac{1}{2}[-\infty + 0 + 2]$$

$$= 1 + 1 = 2$$

$$\sigma^2 = E(x^2) - (E(x))^2 = 2 - 0 = 2$$

3. The probability between 0 and 4

$$P(0 \le x \le 4) = \frac{1}{2}\int_0^4 e^{-x}\,dx \quad \text{Since in the internal 0 to 4, } |x| = x$$

$$= \frac{1}{2}\left[-e^{-x}\right]_0^4$$

$$= \frac{1}{2}\left[1 - e^{-4}\right]$$

$$= \frac{1}{2}\left[1 - 0.0183\right]$$

$$= \frac{1}{2}\left[0.98168\right] \qquad = 0.49084 \qquad \textbf{Ans.}$$

Example 1.88. Let X denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the

1. Discrete probability distribution
2. Expectation
3. Variance

Solution

In the throwing of a pair of fair dice,

The total number of cases are $6 \times 6 = 36$

The minimum number could be 1, 2, 3, 4, 5, 6 i.e. min {a, b}

		Die-1					
Die – 2		1	2	3	4	5	6
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6,4)	6, 5)	(6, 6)

$$\text{Min }\{a, b\} = p(x = 1) = \frac{11}{36}$$

$$\text{Min }\{a, b\} = p(x = 2) = \frac{9}{36}$$

$$\text{Min }\{a, b\} = p(x=3) = \frac{7}{36}$$

$$\text{Min }\{a, b\} = p(x = 4) = \frac{5}{36}$$

$$\text{Min }\{a, b\} = p(x = 5 = \frac{7}{36}$$

$$\text{Min }\{a, b\} = p\,(x = 6) = \frac{1}{36}$$

Discrete probability distribution

$X = x_i$	1	2	3	4	5	6
$P(X=x_i)= f(x_i)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{1}{36}$

2. Mean = Expectation

$$= E(X)$$

$$= \sum x_i f_i$$

$$= 1\times\frac{11}{36} + 2\times\frac{9}{36} + 3\times\frac{7}{36} + 4\times\frac{5}{36} + 5\times\frac{3}{36} + 6\times\frac{1}{36}$$

$$= 0.3055 + 0.5 + 0.58333 + 0.5555 + 0.41666 + 0.166$$

$$= 2.5$$ **Ans.**

3. $\text{Var.}(X) = \sigma^2 = \sum x^2 f_i - \mu^2$

$$= (1)^2\times\frac{11}{36} + (2)^2\frac{9}{36} + (3)^2\times\frac{7}{36} + (4)^2\times\left(\frac{5}{36}\right) + (5)^2\times\left(\frac{3}{36}\right) + (6)^2\left(\frac{1}{36}\right) - (2.5)^2$$

$$= 1\times\frac{11}{36} + 4\times\frac{9}{36} + 9\times\frac{7}{36} + 16\times\frac{5}{36} + 25\times\frac{3}{36} + + \frac{36\times 1}{36} - (2.5)^2$$

$$= 0.30555 + 1 + 1.75 + 2.2222 + 2.08333 + 1 - 6.25$$

$$= 8.3610 - 6.25 = 2.111, \quad \sigma = 1.452$$ **Ans.**

Example 1.89. If X and Y are discrete random variables and k is a constant then prove that

(Nov. 2006 Reg. Set 2)
(Apr. /May 2005 Reg. Set 3)
(Nov. 2008 Supple. Set 2)
(Nov.2009 set 1)

1. E (X+K) = E(X) + K
2. E (X+Y) = E(X) + E(Y)

Solution

1. $$E(X+k)=\frac{\sum f_i(x_i+k)}{\sum f_i}$$

$$=\frac{\sum f_i x_i}{\sum f_i}+k\frac{\sum f_i}{\sum f_i}$$

$= \text{E (X)} + \text{k}$ **Ans.**

2. $$E(X+Y)=\frac{\sum f_i(x_i+y_i)}{\sum f_i}$$

$$=\frac{\sum f_i x_i}{\sum f_i}+\frac{\sum f_i x_i}{\sum f_i}$$

$$=E(X)+\sum(Y)$$ **Ans.**

Example 1.90. For the continuous probability function

$f(x)=kx^2e^{-x}$ When $x \geq 0$, find

(Supple. Feb. 2007 Set 2)
(Reg. Nov. 2006 Set 1)
(Supple Nov. /Dec. 2005. Set 2)
(Supple. Nov 2008.Set.4)

1. k
2. mean
3. variance

Solution 1. k

$$\int_{-\infty}^{\infty} f(x)dx=1$$

$$\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{\infty} f(x)\,dx=1$$

a constant then

:006 Reg. Set 2)
:005 Reg. Set 3)
8 Supple. Set 2)
Nov.2009 set 1)

$$\int_{-\infty}^{0} 0\ dx + \int_{0}^{\infty} kx^2 c^{-k} dx = 1$$

$$\int_{0}^{\infty} kx^2 c^{-x}\ dx = 1$$

$$k\left[x^2 \int e^{-x} dx - \int\left(\frac{d}{dx} x^2 \int e^{-x} dx\right) dx\right]_0^{\infty} = 1$$

$$k\left[x^2\left(-c^{-x}\right) - \int 2x\left(-e -^{x}\right) dx\right]_0^{\infty} = 1$$

$$k\left[-x^2 e^{-x} + 2\left(x \int e^{-x} dx - \int\left(\frac{d}{dx}.x \int e^{-x}\ dx\right) dx\right)\right]_0^{\infty} = 1$$

$$k\left[-x^2 e^{-x} + 2\left(x \int e^{-x} dx - \int\left(\frac{d}{dx}.x \int e^{-x}\ dx\right) dx\right)\right]_0^{\infty} = 1$$

.ns.

$$k\left[-x^2 e^{-x} + 2\left(x\left(-e^{-x}\right) - \int -e^{-x} dx\right)\right]_0^{\infty} = 1$$

$$k\left[-x^2 c^{-x} + 2\left(x\left(-e^{-x}\right) - \int -e^{-x} dx\right)\right]_0^{\infty} = 1$$

$$k\left[-x^2 c^{-x} + 2\left(-xe^{-x} + \left(-e^{-x}\right)\right)\right]_0^{\infty} = 1$$

$$k\left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

Ans.

$$k\left[-\infty^2 e^{-\infty} - 2\infty e^{-\infty} + oc^{-0} - 2.0 - e^{-0} - 2e^{-0}\right] = 1$$

$$k\left[-0 - 2.0 - 2.0 + 0 - 0 - 2\right] = 1$$

$$K = -\frac{1}{2}$$

Feb. 2007 Set 2)
Nov. 2006 Set 1)
Dec. 2005. Set 2)
Nov 2008.Set.4)

2. Mean $= \int_{-\infty}^{\infty} x\ f(x)\ dx$

$$= \int_{-\infty}^{0} x\ f(x)\ dx + \int_{0}^{\infty} x\ f(x)\, dx$$

$$= \int_{-\infty}^{0} x.0\ dx + \int_{0}^{\infty} x\left(k\ x^2 e^{-x}\right) dx$$

$$= \int_{0}^{\infty} \left(k\, x^3 e^{-x}\right) dx$$

$$= k\left[x^3 \int d^{-x} dx - \int\left(\frac{d}{dx} x^3\ \int e^{-x} dx\right) dx\right]_0^{\infty}$$

$$= \left[-x^3 e^{-x} - \int 3x^2\left(-e^{-x}\right) dx\right]_0^{\infty}$$

$$= k\left[-x^3e^{-x} + 3\int x^2e^{-x}dx\right]_0^{\infty}$$

$$= k\left[-x^3e^{-x} + 3\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)\right]_0^{\infty}$$

$$= k\left[-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right]_0^{\infty}$$

$$= k\left[-\infty^3e^{-\infty} - 3\infty^2e^{-\infty} - 6\infty e^{-\infty} - 6e^{-\infty} + 0e^{-0} + 30e^{-0} + 6.0.e^{-0} - 6e^{-0}\right]$$

$$= k[-6]$$

$$= -\frac{1}{2} \times -6 = 3$$

3. Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - (mean)^2$

$$= \int_{-\infty}^{0} x^2 f(x)\, dx + \int_{0}^{\infty} x^2 f(x)dx - (mean)^2$$

$$= \int_{-\infty}^{0} x^2 (o)dx + \int_{0}^{\infty} x^2\left(kx^2e^{-x}\right)dx - (mean)^2$$

$$= \int_{0}^{\infty} kx^4e^{-x} - (mean)^2$$

$$= k\left[x^4 \int e^{-x}dx - \int\left(\frac{d}{dx}x^4 \int e^{-x}dx\right)dx\right]_0^{\infty} - (mean)^2$$

$$= k\left[-x^4e^{-x} - \int 4x^3\left(-e^{-x}\right)dx\right]_0^{\infty} - (mean)^2$$

$$= k\left[-x^4e^{-x} + 4\int x^3e^{-x}dx\right]_0^{\infty} - (mean)^2$$

$$= k\left[-x^4e^{-x} + 4\left(-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right)\right]_0^{\infty} - (mean)^2$$

$$= k\left[-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^{\infty} - (mean)^2$$

$$= k\left[-\infty e^{-\infty} - 4\infty e^{-\infty} - 12\infty e^{-\infty} - 24\infty e^{-\infty} - 24e^{-\infty} + Oe^{-0} - 40e^{-0} - 120e^{-0} - 240e^{-0}\right]$$
$$- (mean)^2$$

$$= k(-24) - (mean)^2$$

$$= -\frac{1}{2}(-24) - (mean)^2$$

$$= -\frac{1}{2}(-24) - (3)^2$$

$$= 12 - 9 \quad = 3$$

Example 1.91. Sa
items, of which 3
sample. Find the e

Solution

A sample of
which 3 are defecti

Total number

Defective iten

Since we have

1. If number of
Then total nur

2. If number of
Then total nur

3. If number of
Then total nur

4. If number of
Then total nur

Thus the prob

$X = x_i$
$P(X=x_i) =$ F

$E(X) = \sum x_i P(x$

Example 1.91. Sample of size 3 is selected at random from a box containing 12 items, of which 3 are defective. Let X denotes the number of defective items in the sample. Find the expected number E(X) of defective items.

(Reg. April. /May 2004 Set 2)

Solution

A sample of size 3 is selected at random from a box containing 12 items of which 3 are defective.

Total number of possible cases = 12_{C_3}

Defective item may be 0, 1, 2, 3

Since we have to select 3 items

1. If number of defective items are zero

   Then total number of favorable cases = $3_{C_0}\ 9_{C3}$

2. If number of defective items are one

   Then total number of favorable cases = $3_{C_1}\ 9_{C2}$

3. If number of defective items are two

   Then total number of favorable cases = $3_{C_2}\ 9_{C1}$

4. If number of defective items are three

   Then total number of favorable cases = $3_{C3}\ 9_{C0}$

Thus the probability distribution of the defectives items

$X = x_i$	0	1	2	3
$P(X=x_i) = P(X)$	$\frac{3_{C_0}\,9_{C_3}}{12_{C_3}}$	$\frac{3_{C_1}\,9_{C_2}}{12_{C_3}}$	$\frac{3_{C_2}\,9_{C_1}}{12_{C_3}}$	$\frac{3_{C_3}\,9_{C_0}}{12_{C_3}}$
	$\frac{1\times 84}{220}$	$\frac{3\times 36}{220}$	$\frac{3\times 9}{220}$	$\frac{1\times 1}{220}$

$$E(X)=\sum x_i P(x_i)=\frac{0\times 84}{220}+\frac{1\times 3\times 36}{220}+\frac{2\times 3\times 9}{220}+\frac{3\times 1\times 1}{220}$$

$$=0.49090+0.24545+0.0136$$

$$=0.79986 \qquad \textbf{Ans.}$$

Example 1.92.

A pair of fair dice is tossed. Let X denotes the maximum of the number appearing i.e. X (a, b) = max (a, b) and let yet Y denote the sum of the numbers appearing i.e. Y (a, b) = a + b. Find the variance and standard deviation of X and Y.

(Reg. April/ May 2004)

Solution

$$P(1) = P(x = 1) = P(1, 1) = \frac{1}{36}$$

X(a, b) = max (a, b)

		Die - I					
		1	**2**	**3**	**4**	**5**	**6**
Die - II	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6,4)	6, 5)	(6, 6)

$$P(1) = P(1, 1) = \frac{1}{36}$$

$$P(2) = P[(2, 1)(2, 2)(1, 2)] = \frac{3}{36}$$

$$P(3) = P[(1, 3)(3, 1)(2, 3)(3, 2)(3, 3)] = \frac{5}{36}$$

$$P(4) = P[(1, 4)(4, 1)(2, 4)(4, 2)(3, 4)(4, 3)(4, 4)] = \frac{7}{36}$$

$$P(5) = P[(1, 5)(5, 1)(2, 5)(5, 2)(3, 5)(5, 3)(4, 5)(5, 4)(5, 5)] = \frac{9}{36}$$

$$P(6) = P[(1,6)(6,1)(2,6)(6,2)(3,6)(6,3)(4,6)(6,4)(5,6)(6,5)(6,6) = \frac{11}{36}$$

Thus the required discrete probability distribution

$X = x_i$	1	2	3	4	5	6
$P(X = x_i) = f(x_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Mean = Expectation = E(X) = $\sum x_i f_i$

$$= 1\times\frac{1}{36}+2\times\frac{3}{36}+3\times\frac{5}{36}+4\times\frac{7}{36}+5\times\frac{9}{36}+6\times\frac{11}{36}$$

$$= 0.02777 + 0.1666 + 0.41666 + 0.7777 + 1.25 + 1.833$$

$$= 4.47207$$ **Ans.**

Variance $\sigma^2 = \sum x^2 f_i - \mu^2$

$$= 1\times\frac{1}{36}+4\times\frac{3}{36}+9\times\frac{5}{36}+16\times\frac{7}{36}+25\times\frac{9}{36}+36\times\frac{11}{36}-(4.47)^2$$

$$= 0.0277 + 0.3333 + 1.25 + 3.111 + 6.25 + 11 - 19.9994$$

$$= 21.97207 - 19.9994$$

$$= 1.97267$$

Standard deviation σ = 1.40451

II part

Y (a, b) = a + b

The total number of cases are 6 × 6 = 36

The addition could be {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12}

Y (a, b) = a + b

	Die-1						
		1	2	3	4	5	6
Die – 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

$$P(2) = P(1, 1) = \frac{1}{36}$$

$$P(3) = P[(1, 2)(2, 1)] = \frac{2}{36}$$

$$P(4) = P[(3, 1)(2, 2)(1, 3)] = \frac{3}{36}$$

$$P(5) = P[(4, 1)(3, 2)(2, 3)(4, 1)] = \frac{4}{36}$$

$$P(6) = P[(5, 1)(4, 2)(3, 3)(2, 4)(1, 5)] = \frac{5}{36}$$

$$P(7) = P[(6, 1)(5, 2)(4, 3)(3, 4)(2, 5)(1, 6)] = \frac{6}{36}$$

$$P(8) = P[(6, 2)(5, 3)(4, 4)(3, 5)(2, 6)] = \frac{5}{36}$$

$$P(9) = P[(6, 3)(5, 4)(4, 5)(3, 6)] = \frac{4}{36}$$

$$P(10) = P[(6, 4)(5, 5)(4, 6)] = \frac{3}{36}$$

$$P(11) = P[(6, 5)(5, 6)] = \frac{2}{36}$$

$$P(12) = P[(6, 6)] = \frac{1}{36}$$

Thus the required discrete probability distribution

$Y = y_I$	2	3	4	5	6	7	8	9	10	11	12
$P(Y = y_i) = f(y_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean = Expectation = $E(y) = \sum y_i f_i$

$$= 2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{36}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}$$
$$+11\times\frac{2}{36}+12\times\frac{1}{36}$$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36}$$

$$= \frac{252}{36} = 7$$ **Ans.**

$$\text{Variance }(Y) = \sum y_i^2 f_i - \mu^2$$

$$= 4\times\frac{1}{36}+9\times\frac{2}{36}+16\times\frac{3}{36}+25\times\frac{4}{36}+36\times\frac{5}{36}+49\times\frac{6}{36}+64\times\frac{5}{36}$$

$= \frac{4+18+}{}$

Variance σ^2

Standard de

Example 1.93.
if 2 tails occur, I
heads occurs. Fin

Solution

Let X = D.R
The sample
S = {H, T
= {HHI

Probability o

Probability o

Probability c

Probability o

Describe pro

Expected val

=

=

$$+81\times\frac{4}{36}+100\times\frac{3}{36}+121\times\frac{2}{36}+144\times\frac{1}{36}=-49$$

$$=\frac{4+18+48+100+180+294+320+324+300+242+140}{36}$$

$$\text{Variance } \sigma^2=\frac{1974-49}{36}=54.8333-49=5.8333$$

Standard deviation $\sigma = 2.4152$ **Ans.**

Example 1.93. A player tosses 3 fair coins. He wins Rs. 800, if 3 tail occur, Rs.500 if 2 tails occur, Rs.300 if one tail occurs. On the other hand, he loses Rs.1000 if 3 heads occurs. Find the value of the game to the player. Is it favorable?

(Reg. April/May 2004 Set 1)

Solution

Let X = D.R.V = number of tails occurs in 3 tosses of a fair coins.

The sample space S is

$$S = \{H, T\} \times \{H, T\} \times \{H, T\}$$
$$= \{HHH, HHT, HTH, HTT, THH, TTH, TTT, THT\}$$

Probability of all 3 tails = $P(X = 3) = \frac{1}{8}$

Probability of all 3 heads = $P(X = 0) = \frac{1}{8}$

Probability of all 2 tails = $P(X = 2) = \frac{3}{8}$

Probability of all 1 tails = $P(X = 1) = \frac{3}{8}$

Describe probability distribution is

$X = x_i$	0	1	2	3
$P(X = xi)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Expected value of the game

$$=800\times\frac{1}{8}+500\times\frac{3}{8}+300\times\frac{3}{8}-1000\times\frac{1}{8}$$

$$=\frac{800+1500+900-1000}{8}$$

$$= \frac{2200}{8}$$

$= 275$ rupees **Ans.**

Since E > 0, game is favorable to the player.

Example 1.94. Let F(X) be the distribution function of a random variable X given by

(Reg.April/ May 2005 Set 4)

(Supple. Nov 2008.Set.3)

$$F(X) = cx^3 \text{ when } 0 \le x < 3$$
$$= 1 \text{ when } x \ge 3$$
$$= 0 \text{ when } x < 0$$

If $P(X = 3) = 0$ determine

1. c
2. mean
3. P(X>1)

Solution

We know that

$$F'(x) = f(x)$$

$$f(x) = \begin{cases} 3cx^2 = \dfrac{x^2}{9} & \text{if } 0 \le x < 3 \\ 0 & \text{if } x \ge 3 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\int_{0}^{3} 3cx^2 dx = 1$$

$$3c\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$3c\left[\frac{27}{3} - 0\right] = 1$$

$$27\,c = 1$$

$$c = \frac{1}{27}$$

2. Mean

$$= \int_{-\infty}^{\infty} xf(\ldots$$

$$= \int_{-\infty}^{0} x\,f\ldots$$

$$= \int_{-\infty}^{0} x.0\ldots$$

$$= 3c\int_{0}^{3} x^3\ldots$$

$$= 3c\left[\frac{x^4}{4}\ldots\right.$$

$$= 3 \times \frac{1}{27} \times \ldots$$

$$= \frac{9}{4}$$

3. P(x>1) = $\int \ldots$

$$= \int^{3} \ldots$$

$$= \int^{3} \ldots$$

$$= 3c\ldots$$

$$= 3c\ldots$$

$$= 3 \times \ldots$$

$$= \frac{26}{27}\ldots$$

le X given by
ay 2005 Set 4)
ov 2008.Set.3)

$$c = \frac{1}{27}$$ **Ans.**

2. Mean

$$= \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{0} x\, f(x) dx + \int_{0}^{3} x\; f(x) dx + \int_{3}^{\infty} x\; f(x) dx$$

$$= \int_{-\infty}^{0} x.0\, dx + \int_{0}^{3} k\, 3cx^2 dx + \int_{3}^{\infty} k.0\, dx$$

$$= 3c \int_{0}^{3} x^3 dx$$

$$= 3c \left[\frac{x^4}{4}\right]_0^3$$

$$= 3 \times \frac{1}{27} \times \frac{81}{4}$$

$$= \frac{9}{4}$$ **Ans.**

3. $$P(x>1) = \int_{1}^{\infty} f(x)\, dx$$

$$= \int_{1}^{3} f(x) dx + \int_{3}^{\infty} f(x)\, dx$$

$$= \int_{1}^{3} 3cx^2 dx + \int_{3}^{\infty} 0\, dx$$

$$= 3c \left[\frac{x^3}{3}\right]_1^3$$

$$= 3c \times \left(\frac{27}{3} - \frac{1}{3}\right)$$

$$= 3 \times \frac{1}{27} \times \frac{26}{3}$$

$$= \frac{26}{27}$$ **Ans.**

Example 1.95.

If a random variable has the probability density

(JNTU 2001)

(Supple. Nov 2008.Set.1)

$$f(x) = \begin{cases} 2e^{-2x} & for\ x > 0 \\ 0 & for\ x \le 0 \end{cases}$$

Find the probability that it will take on a value

1. Between 1 and 3
2. Greater than 0.5

Solution

1. Between 1 and 3

$$\int_1^3 2e^{-2x}dx = 2\left[\frac{e^{-2x}}{-2}\right]_1^3$$

$$= -\left[e^{-2x}\right]_1^3$$

$$= -\left[e^{-2\times 3} - e^{-2\times 1}\right]$$

$$= -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - 3^{-6}$$

$$= 0.1353 - 0.002478$$

$$= 0.132822$$ **Ans.**

2. Greater than 0.5

$$\int_{0.5}^{\infty} 2e^{-2x}dx = 2\int_{0.5}^{\infty} e^{-2x}dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty}$$

$$= -\left[e^{-2\infty} - e^{-2\times 0.5}\right]$$

$$= -\left[e^{-\infty} - e^{-1}\right]$$

$$= e^{-1} - e^{-\infty}$$

$$= 0.3678 - 0

(JNTU 2001)
(Nov 2008.Set.1)

Example 1.96. Probability density function of a random variable

(JNTU 2004)

$$f(x) = \begin{cases} \frac{1}{2}\sin x & 0 \le x \le \pi \\ 0 & \textit{else where} \end{cases}$$

Find the Mean Mode and Median for the distribution and also find the probability between 0 and $\pi/2$.

1. Mean = $\int_{-\infty}^{\infty} x\, f(x)dx$

$$= \int_{-\infty}^{0} x\ f(x)dx + \int_{0}^{\pi} xf(x)dx + \int_{\pi}^{\infty} x\, f(x)dx$$

$$= \int_{-\infty}^{0} x\ 0dx + \int_{0}^{\pi} x\frac{1}{2}\sin x\ dx + \int_{\pi}^{\infty} x.0\,dx$$

$$= \frac{1}{2}\int_{0}^{\pi} x\sin x\ dx$$

$$= \frac{1}{2}\left[x\int \sin x\ dx - \int\left(\frac{d}{dx}x\int \sin x\,dx\right)dx\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[x(-\cos x) - \int(-\cos x)dx\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[-x\cos x + \sin x\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[-\pi\cos\pi + \sin\pi + 0 - \sin o\right]$$

$$= \frac{1}{2}\left[-\pi(-1)\right] = \frac{\pi}{2}$$ **Ans.**

2. Mode

We know that mode is the value of X for which f(x) is maximum

f'(x) = 0 and f''(x) is negative at that value

$$f'(x) = \frac{1}{2}\cos x$$

$$f''(x) = -\frac{1}{2}\ \sin x$$

$$= -\frac{1}{2} < 0 \ \ when \ x = \frac{\pi}{2}$$

$$\boxed{\text{Mode} = \frac{\pi}{2}}$$

Suppose M is the median then

$$\int_0^m f(x)dx = \int_m^\pi f(x)dx = \frac{1}{2}$$

$$= \int_m^\pi \frac{1}{2}\sin x \ dx = \frac{1}{2}$$

$$= \frac{1}{2}[-\cos x]_0^M = \frac{1}{2}$$

$$= \frac{1}{2}(-\cos M + \cos o) = \frac{1}{2}$$

$$= \frac{1}{2} - \frac{\cos M}{2} = \frac{1}{2}$$

(cos$M$ should be zero)

$$\boxed{\text{M} = \frac{\lambda}{2}}$$

Ans.

Example 1.97. A continuous random variable has the p.d.f.

$$f(x) = \begin{cases} kxe^{-\lambda x} & if \ \ x \geq 0 \quad \lambda > 0 \\ 0 & otherwise \end{cases}$$

Determine the constant k, find mean and variance. **(JNTU 2003 S)**

Solution.

Given that

$$f(x) = \begin{cases} kxe^{-\lambda x} & if \ \ x \geq 0 \quad \lambda > 0 \\ 0 & other \ wise \end{cases}$$

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} f(x)dx + \int_0^\infty f(x) = 1$$

$$\int_{-\infty}^{0} 0\ dx + \int_{0}^{\infty} k\, xe^{-\lambda x} dx = 1$$

$$k\int_{0}^{\infty} xe^{-\lambda x} dx = 1$$

$$k\left[k\int e^{-\lambda x} - \int\left(\frac{d}{dx}x.\int e^{-\lambda x}\right)dx\right]_0^{\infty} = 1$$

$$k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - \int\frac{e^{-\lambda x}}{-\lambda}dx\right]_0^{\infty} = 1$$

$$k\left[\frac{-xe^{-\lambda x}}{\lambda} + \int\frac{e^{-\lambda x}}{+\lambda}\ dx\right]_0^{\infty} = 1$$

$$k\left[\frac{-xe^{-\lambda x}}{\lambda} + \frac{e^{-\lambda x}}{(+\lambda)(-\lambda)}\right]_0^{\infty} = 1$$

$$k\left[\frac{-\infty e^{-\infty}}{\lambda} + \frac{e^{-\infty}}{\lambda^2} + 0 + \frac{e^{0}}{\lambda^2}\right] = 1$$

$$k\frac{1}{\lambda^2} = 1$$

$$\boxed{k = \lambda^2}$$

Ans.

JNTU 2003 S)

$$\text{Mean} = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$= \int_{-\infty}^{0} x\ f(x)\,dx + \int_{0}^{\infty} x\ f(x)\,dx$$

$$= \int_{-\infty}^{0} x\ 0\ dx + \int_{0}^{\infty} x.k\,xe^{-\lambda x} dx$$

$$= k\int_{0}^{\infty} x^2 e^{-\lambda x} dx$$

$$= k\left[x^2\int e^{-\lambda x}dx - \int\left(\frac{d}{dx}x^2\int e^{-\lambda x}dx\right)dx\right]_0^{\infty}$$

$$= k\left[x^2\frac{e^{-\lambda x}}{-\lambda} - \int\left(2x\frac{e^{-\lambda x}}{-\lambda}\right)dx\right]_0^{\infty}$$

$$= k\left[\frac{x^2\ e^{-\lambda x}}{-\lambda} + 2\int \frac{x e^{-\lambda x}}{\lambda}\, dx\right]_0^\infty$$

$$= k\left[\frac{x^2 e^{-\lambda x}}{-\lambda} + \frac{2}{\lambda}\left\{x\int e^{-\lambda x} dx - \int\left(\frac{d}{dx} x \int e^{-\lambda x} dx\right)\right]_0^\infty$$

$$= k\left[\frac{x^2 e^{-\lambda x}}{-\lambda} + \frac{2}{\lambda}\left\{x\frac{e^{-\lambda x}}{-\lambda} - \int\frac{e^{-\lambda x}}{-\lambda} dx\right\}\right]_0^\infty$$

$$= k\left[\frac{x^2 e^{-\lambda x}}{-\lambda} + \frac{2}{\lambda}\left\{\frac{xc^{-\lambda x}}{-\lambda} + \frac{1}{\lambda}\int e^{-\lambda x} dx\right\}\right]_0^\infty$$

$$= k\left[\frac{x^2 e^{-\lambda x}}{-\lambda} + \frac{2xe^{-\lambda x}}{-\lambda^2} + \frac{2}{\lambda^2}\frac{e^{-\lambda x}}{(-\lambda)}\right]_0^\infty$$

$$= k\left[\frac{x^2 e^{-\lambda x}}{-\lambda} - \frac{2xe^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3}\right]_0^\infty$$

$$= k\left[\frac{2e^{-0}}{\lambda^3}\right] = \frac{2\lambda^2}{\lambda^3} = \frac{2}{\lambda}$$

$$\boxed{\text{Mean} = \frac{2}{\lambda}}$$ **Ans.**

$$\text{Variance} = \int_{-\infty}^{\infty} x^2 f(x)dx - (mean)^2$$

$$= \int_{-\infty}^{0} x^2 f(x)dx + \int_0^\infty x^2 f(x)dx - (mean)^2$$

$$= \int_0^\infty x^2 f(x)dx - (mean)^2$$

$$= \int_0^\infty x^2 kxe^{-\lambda x} dx - (mean)^2$$

$$= k\int_0^\infty x^3 e^{-\lambda x} dx - (mean)^2$$

$$= \lambda^2\left[\frac{x^3 e^{-\lambda x}}{-\lambda} - \frac{3x^2 e^{-\lambda x}}{\lambda^2} - \frac{6xe^{-\lambda x}}{\lambda^3} - \frac{6e^{-\lambda x}}{\lambda^4}\right]_0^\infty - \left(\frac{2}{\lambda}\right)$$

$$= \lambda^2 \times \frac{6}{\lambda^4} - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$\boxed{\text{Variance} = \frac{2}{\lambda^2}}$$ **Ans.**

Example 1.98. Continuous random variable X is defined by **(JNTU 2003 S)**

$$f(x) = \begin{cases} \dfrac{(3+x)^2}{16} & \text{if } -3 \le x < -1 \\ \dfrac{(6-2x^2)}{16} & \text{if } -1 \le x < 1 \\ \dfrac{(3-x)^2}{16} & \text{if } 1 \le x \le 3 \\ 0 & \text{if else where} \end{cases}$$

Verify that f(x) is density function and find also the mean of X.

Solution

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{-3} f(x)dx + \int_{-3}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx$$

$$\text{L.H.S.} = \int_{-\infty}^{-3} o\,dx + \int_{-3}^{-1} \frac{(3+x)^2}{16}\,dx + \int_{-1}^{1} \frac{(6-2x^2)}{16}\,dx + \int_{1}^{3} \frac{(3-x)^2}{16}\,dx + \int_{3}^{\infty} o\,dx$$

$$= \int_{-3}^{-1} \frac{(9+x^2+6x)}{16}dx + \int_{-1}^{1}\left(\frac{6-2x^2}{16}\right)dx + \int_{1}^{3}\left(\frac{9+x^2-6x}{16}\right)dx$$

$$= \int_{-3}^{-1} \frac{9}{16}dx + \int_{-3}^{-1} \frac{x^2}{16}dx + \int_{-3}^{-1} \frac{6}{16}x\,dx + \int_{-1}^{1} \frac{6}{16}\,dx - \int_{-1}^{1} \frac{2}{16}x^2\,dx$$

$$+ \int_{1}^{3} \frac{9}{16}dx + \int_{1}^{3} \frac{x^2}{16}dx - \int_{1}^{3} \frac{6}{16}x\,dx$$

$$= \frac{9}{16}[x]_{-3}^{-1} + \frac{1}{16}\left[\frac{x^3}{3}\right]_{-3}^{-1} + \frac{6}{16}\left[\frac{x^2}{2}\right]_{-3}^{-1} + \frac{6}{16}[x]_{-1}^{1} - \frac{2}{16}\left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{9}{16}[x]_{1}^{3}$$

$$+\frac{1}{16}\left[\frac{x^3}{3}\right]_1^3-\frac{6}{16}\left[\frac{x^2}{2}\right]_1^3$$

$$=\frac{9}{16}\times[-1+3]+\frac{1}{16\times 3}[-1+27]+\frac{6}{16\times 2}[1-9]+\frac{6}{16}[1+1]-\frac{2}{16\times 3}[1+1]$$

$$+\frac{9}{16}[3-1]+\frac{1}{16\times 3}[27-1]-\frac{6}{16\times 2}[9-1]$$

$$=\frac{9}{8}+\frac{13}{24}-\frac{3}{2}+\frac{3}{4}-\frac{1}{12}+\frac{9}{8}+\frac{13}{24}-\frac{24}{16}\ or\ \frac{12}{8}$$

$$=\frac{27+13-36+18-2+27+13-36}{24}$$

$$=\frac{24}{24}$$

$$=1$$

$\Rightarrow$ f(x) is density function **Ans**

$$\text{Mean} = \int_{-\infty}^{\infty} xf(x)dx$$

$$=\int_{-3}^{-1}x\,f(x)dx+\int_{-1}^{1}x\,f(x)dx+\int_{1}^{3}x\,f(x)dx$$

$$=\int_{-3}^{-1}\frac{(3+x)^2}{16}x\,dx+\int_{-1}^{1}\frac{(6-2x^2)}{16}x\,dx+\int_{1}^{3}\frac{(3-x)^2}{16}x\,dx$$

$$=\int_{-3}^{-1}\left(\frac{9x+6x^2+x^3}{16}\right)dx+\int_{1}^{3}\left(\frac{9x-6x^2+x^3}{16}\right)dx$$

$$=0$$

We know that for any odd function

$$\int_{-a}^{a} f(x)dx=0 \qquad f(x)=\left(\frac{6-2x^2}{16}\right)x$$

$$\int_{-1}^{1} x\left(\frac{6-2x^2}{16}\right)dx=0 \qquad f(-x)=\left(\frac{6-2x^2}{16}\right)(-x)$$

Mean = 0

$$f(x)=-f(-x)$$

Example 1.99.

If a ball is c
find the probabili

a. Less tl
b. Even r
c. Odd nı
d. Prime
e. Find tl

Solution

Given that a

Since each t
distribution is disc

a. The probabili

$P(x<5)=$

b. The probabili

Even number

Probability of

c. The probabilit

Odd numbers

Probability of

d. The probabilit

Prime number

Probability of

e. Mean and the

Example 1.99.

If a ball is drawn from a box containing 10 balls numbered 1 to 10 inclusive find the probability that the number X drawn is

a. Less than 5

b. Even number

c. Odd number

d. Prime number

e. Find the mean and variance of the random variable X

Solution

Given that a box containing 10 balls numbered 1 to 10 inclusive.

Since each balls has the same probability for being drawn, the probability distribution is discrete uniform distribution given by

$$f(x) = \frac{1}{10} \quad for \quad x = 1, 2, 3.......10$$

a. The probability that the number X drawn is less than 5

$$P(x < 5) = \sum_{x=0}^{4} P(x) = \sum_{x=0}^{4} \frac{1}{10} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

b. The probability that the number X drawn is even number

Even numbers are 2, 4, 6, 8, 10 with the probability $\frac{1}{10}$ each

$$\text{Probability of even number} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}$$

c. The probability that the number X drawn is odd number

Odd numbers are 1, 3, 5, 7, 9 with the probability $\frac{1}{10}$ each.

$$\text{Probability of odd number} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}$$

d. The probability that the number X drawn is prime number

Prime numbers are 2, 3, 5, 7 with the probability $\frac{1}{10}$ each

$$\text{Probability of prime number} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

e. Mean and the variance of the random variable X

$$\text{Mean} = E(x) = \sum_{i=1}^{10} x_i P(x_i)$$

$$= \sum_{x=1}^{10} x\ P(x)$$

$$= \frac{1}{10}(1+2+3+4+5+6+7+8+9+10)$$

$$\text{Mean } \mu = \frac{1}{10} \times 55 = 5.5$$

$$\text{Variance} = \sum_{x=1}^{10} (x-\mu)^2\ P(x)$$

$$= \frac{1}{10}[(1-5.5)^2 + (2-5.5)^2 + (3-5.5)^2 + (4-5.5)^2 + (5-5.5)^2 + (6-5.5)^2$$
$$+ (7-5.5)^2 + (8-5.5)^2] + (9-5.5)^2 + (10-5.5)^2]$$

$$= \frac{1}{10}[20.25 + 12.25 + 6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25 + 12.25 + 20.25]$$

$$= \frac{1}{10}[82.5] = 8.25$$ **Ans.**

Example 1.100.

Find the probability that a card drawn at random from 20 cards numbered 1 to 20

(a) Is Prime (b) Ends in the digit 2

Solution

Given that

A card drawn at random from 20 cards numbered 1 to 20

Since each card has the same probability for being drawn. The probability distribution is discrete uniform distribution given by

$$f(x) = \frac{1}{20} \ for \ x = 1, 2, 3......20$$

a. The probability that a card drawn at random from 20 cards numbered 1 to 20 is prime.

Prime numbers are

(2, 3, 5, 7, 11, 13, 17, 19) with the probability $\frac{1}{20}$ each.

Probability of prime number

$$= \frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}=\frac{8}{20}$$

b. The probability that a card drawn at random from 20 cards numbered 1 to 20 is ends in the digit 2

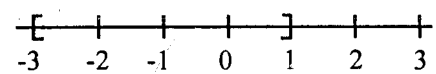
Numbers are (2, 12)

Probability of the numbers ends in the digit 2

$$= \frac{1}{20}+\frac{1}{20}=\frac{2}{20}=\frac{1}{10}$$

Example 1.101.

If X is uniformly distributed in $-3 \le x \le 3$ find



(a) $P(x < 1)$

Solution

Given that x is uniformly distributed in $-3 \le X \le 3$

a. $P(x < 1)$

Since $(x < 1)$ it lies in the internal [-3, 1] of length 4.

$$P(c \le x \le d) = \int_c^d f(x)\ dx$$

$$= \int_c^d \frac{1}{b-a} dx$$

$$= \frac{d-c}{b-a}$$

$$P(x<1)=\frac{1-(-3)}{3-(-3)}=\frac{1+3}{3+3}=\frac{4}{6}=\frac{2}{3}$$ **Ans.**

Example 1.102.

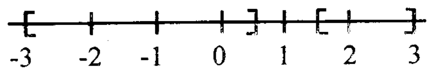
If X is uniformly distributed in $-3 \le x \le 3$ find $P\left(|x-1| \ge \frac{1}{2}\right)$

Solution

Given that X is uniformly distributed in $-3 \le x \le 3$

If $|x - 1| \ge \frac{1}{2}$ then

$x \le \frac{1}{2}$ *and* $x \ge \frac{3}{2}$ i.e. x lies in the intervals



$$\left[-3, \frac{1}{2}\right] \text{ and } \left[\frac{3}{2}, 3\right]$$

$$P\left(|x-1| \ge \frac{1}{2}\right) = P\left(-3 \le x \le \frac{1}{2}\right) + P\left(\frac{3}{2}, \le x \le 3\right)$$

$$= \frac{d_1 - c_1}{b_1 - a_1} + \frac{d_2 - c_2}{b_2 - a_2}$$

$$= \frac{\frac{1}{2} - (-3)}{3 - (-3)} + \frac{3 - \frac{3}{2}}{3 - (-3)}$$

$$= \frac{\frac{1}{2} + 3}{6} + \frac{3 - \frac{3}{2}}{6}$$

$$= \frac{7}{12} + \frac{3}{12}$$

$$= \frac{10}{12} = \frac{5}{6}$$ **Ans.**

Example 1.103.

Random variable X has the following probability distribution

X = x	-2	-1	0	1	2	3
$f(x_i) = P(X = x_i)$	0.2	0.1	k	2k	0.4	2k

Find (1) k (2) $P(X \ge 2)$ (3) $P(X<2)$ (4) $P(-2 < x < 3)$

Solution

We know that

1. In the discrete probability distribution

$\sum_{x_i} f$

0.2 + 0.1

0.7 + 5k

5k = 0.3

k = 0.06

2. $P(X \ge 2)$

3. $P(X < 2)$

4. P(-2 < x <

= P (

= 0.1

= 0.5

= 0.5

= 0.6

Example 1.104

A random

$f(x) = \{$

Find

1. P(

2. P (

3. P (

Solution

1.

$$\sum_{x_i} f(x_i) = 1$$

$0.2 + 0.1 + k + 2k + 0.4 + 2k = 1$

$0.7 + 5k = 1$

$5k = 0.3$

$k = 0.06$ **Ans.**

2. $P(X \geq 2) = P(X = 2) + P(X = 3)$

$= 0.4 + 2k$

$= 0.4 + 2 \times 0.06$

$= 0.4 + 0.12$

$= 0.52$ **Ans.**

3. $P(X < 2) = 1\ P(X \geq 2)$

$= 1 - 0.52$

$= 0.48$ **Ans.**

4. $P(-2 < x < 3)$

$= P(X = 1) + P(X = 0) + P(X = 1) + P(X = 2)$

$= 0.1 + k + 2k + 0.4$

$= 0.5 + 3k$

$= 0.5 + 0.18$

$= 0.68$ **Ans.**

Example 1.104.

A random variable X has the density function

$$f(x) = \begin{cases} \frac{1}{2} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Find

1. $P(x < 1)$
2. $P(|x| > 2)$
3. $P(2x + 3 > 6)$

Solution

1. $$P(x < 1) = \int_{-3}^{1} f(x)\,dx = \int_{-3}^{1} \frac{1}{2}\,dx = \frac{1}{2}[x]_{-3}^{1}$$

$$= \frac{1}{2}[1 - (-3)] = \frac{1}{2}(1 + 3) = 2$$ **Ans.**

2. $$P(|x|>2)=P(x>2 \quad and \quad x<-2)$$

$$=\int_{2}^{3} f(x)\,dx+\int_{-3}^{-2} f(x)\,dx$$

$$=\int_{2}^{3}\frac{1}{2}\,dx+\int_{-3}^{-2}\frac{1}{2}\,dx$$

$$=\frac{1}{2}[x]_{2}^{3}+\frac{1}{2}[x]_{-3}^{-2}$$

$$=\frac{1}{2}[3-2]+\frac{1}{2}[-2-(-3)]$$

$$=\frac{1}{2}(1)+\frac{1}{2}(1)=1$$ **Ans.**

3. $$P(2x+3>6)=P(2x>3)=P\left(x>\frac{3}{2}\right)$$

$$=\int_{3/2}^{3} f(x)\,dx=\int_{3/2}^{3}\frac{1}{2}\,dx$$

$$=\frac{1}{2}[x]_{3/2}^{3}=\frac{1}{2}\left[3-\frac{3}{2}\right]$$

$$=\frac{1}{2}\left[\frac{6-3}{2}\right]=\frac{3}{4}$$ **Ans.**

Example 1.105. If the probability function of a random variable is given by

$$f(x)=\begin{cases}k(x^2-1) & 0<x<2\\ 0 & \text{else where}\end{cases}$$

Find

1. k
2. F(x)

Solution

1. We know that $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_{0}^{2} k(x^2-1)\,dx=1$$

$$k\left[\frac{x^3}{3}-x\right]$$

$$k\left[\frac{8}{3}-2-\right.$$

$$k\left[\frac{8-6}{3}\right]=$$

$$k\ \frac{2}{3}=1$$

$$k=\frac{3}{2}$$

We know that for
function f(x), the c

$$F(x)=\int_{-\infty}^{x}$$

$$=\left\{\frac{3}{2}\right.$$

Example 1.106. A
between X = 3 and

Solution

1. We know that

$$k\left[\frac{x^3}{3}-x\right]_0^2=1$$

$$k\left[\frac{8}{3}-2-o\right]=1$$

$$k\left[\frac{8-6}{3}\right]=1$$

$$k\ \frac{2}{3}=1$$

$$k=\frac{3}{2}$$

Ans.

We know that for the continuous random variable X with probability distribution function f(x), the cumulative distribution function F(x)

$$F(x)=\int_{-\infty}^{x} f(x)\,dx \quad -\infty < x < \infty$$

$$=\begin{cases}\frac{3}{2}\left(\frac{x^3}{3}-x\right) & 0<x<2\\ 0 & else\,where\end{cases}$$

Example 1.106. A continuous random variable X that can assume any value between X = 3 and X = 6 has a density function given by $f(x) = k(1 + x^2)$. Find

1. k 2. P (x < 5)

Solution

1. We know that for continuous random variable

$$\int_{-\infty}^{\infty} f(x)\ dx=1$$

$$\int_3^6 k\left(1+x^2\right)dx=1$$

$$k\int_3^6\left(1+x^2\right)dx=1$$

$$k\left[x+\frac{x^3}{3}\right]_3^6=1$$

$$k\left[6+\frac{216}{3}-3-\frac{27}{3}\right]=1$$

$$k\left[\frac{18+216-9-27}{3}\right]=1$$

$$k\left(\frac{198}{3}\right)=1$$

$$k[66]=1$$

K = 0.01515 **Ans.**

2. P (x < 5)

$$P(x<5)=\int_3^5 f(x)\,dx$$

$$=\int_3^5 k(1+x^2)dx$$

$$=k\int_3^5 (1+x^2)\,dx$$

$$=k\left[x+\frac{x^3}{3}\right]_3^5$$

$$=k\left[5+\frac{125}{3}-3-\frac{27}{3}\right]$$

$$=\frac{1}{66}\left[\frac{15+125-9-27}{3}\right]$$

$$=\frac{1}{66}\times\frac{104}{3}$$

$$=0.5252$$ **Ans.**

Example 1.107. The cumulative distribution function of X

$$F(x)=\begin{cases}0 & x>0\\ x^2+1 & 0\le x\le 1\\ 1 & x>1\end{cases}$$

Find

2. f(x)
3. P (0.5 < x ≤ 0.8
4. P (x > 0.8) and P(x ≤ 0.5)

Solution

1. We have

2. P (0.5 < x ≤ (

3. P (x > 0.8) a

P(x >

P (x ≤ 0..

P (x ≤

ns.

Solution

1. We have $f(x) = \dfrac{d}{dx}(F(x))$

$$= \begin{cases} 2x & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

2. P (0.5 < x ≤ 0.8) $= \int_{0.5}^{0.8} f(x)dx$

$$= \int_{0.5}^{0.8} 2x \; dx$$

$$= \left[2\frac{x^2}{2}\right]_{0.5}^{0.8}$$

$$= (0.8)^2 - (0.5)^2$$

$$= 0.64 - 0.25$$

$$= 0.39$$

Ans.

3. P (x > 0.8) and P(x ≤ 0.5)

$$P(x > 0.8) = \int_{0.8}^{1} f(x)dx$$

$$= \int_{0.8}^{1} 2x \; dx$$

$$= \left[2\frac{x^2}{2}\right]_{0.8}^{1}$$

$$= 1 - (0.8)^2$$

$$= 1 - 0.64$$

$$= 0.36$$

Ans.

$$P(x \le 0.5) = \int_{0}^{0.5} f(x) \; dx$$

$$= \int_{0}^{0.5} 2x \; dx$$

$$= \left[2\frac{x^2}{2}\right]_{0}^{0.5}$$

$$= (0.5)^2 - 0$$

$$P(x \le 0.5) = 0.25$$

Ans.

Example 1.108. Define random variable discrete probability distribution continuous probability distribution and cumulative distribution. Give an example of each.

Solution

As define on page no. 1.8,1.9.

Example 1.109. Define discrete and continuous probability distributions with an example.

Solution

As define on page no. 1.9,1.12.

Example 1.110. A random variable gives measurements X between 0 and 1 with probability function **(JNTU 2003)**

$$f(x) = \begin{cases} 12x^3 - 21x^2 + 10x & 0 \le x \le 1 \\ 0 & else\ where \end{cases}$$

1. Find P (x ≤ $\frac{1}{2}$) and P(x > $\frac{1}{2}$)

2. Find a number k such that p (x ≤ k) = $\frac{1}{2}$

Solution

We know that far the probability density function f(x)

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} o\,dx + \int_{0}^{1} \left(12x^3 - 21x^2 + 10x\right)dx + \int_{1}^{\infty} o\ dx = 1$$

$$\left[12\frac{x^4}{4} - 21\frac{x^3}{3} + 10\frac{x^2}{2}\right]_0^1 = 1$$

$$\left[\frac{12\times1}{4} - \frac{21\times1}{3} + \frac{10\times1}{2} - 0\right] = 1$$

$$[3 - 7 + 5] = 1$$

$$1 = 1$$

So f(x) is a probability density function

ion continuous
of each.

utions with an

n 0 and 1 with
(JNTU 2003)

1. $P\left(x \le \frac{1}{2}\right)$ *and* $P\left(x > \frac{1}{2}\right)$

$$P\left(x \le \frac{1}{2}\right) = \int_{-\infty}^{1/2} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{1/2} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1/2} \left(12x^3 - 21x^2 + 10x\right)dx$$

$$= \left[12\frac{x^4}{4} - 21\frac{x^3}{3} + 10\frac{x^2}{2}\right]_0^{\frac{1}{2}}$$

$$= \left[\frac{12}{4} \times \frac{1}{16} - \frac{21}{3} \times \frac{1}{8} + \frac{10}{2} \times \frac{1}{4} - 0\right]$$

$$= \frac{3}{16} - \frac{7}{8} + \frac{5}{4}$$

$$= \frac{3 - 14 + 20}{16}$$

$$= \frac{9}{16}$$

We know that $P\left(x > \frac{1}{2}\right) = 1 - P\left(x \le \frac{1}{2}\right)$

$$= 1 - \frac{9}{16}$$

$$= \frac{16 - 9}{16}$$

$$= \frac{7}{16}$$

3. Find a number k such that $P(x \le k) = \frac{1}{2}$

$$P(x \le k) = \frac{1}{2}$$

$$\int_0^k f(x)\,dx = \frac{1}{2}$$

$$\int_0^k \left(12x^3 - 21x^2 + 10x\right)dx = \frac{1}{2}$$

$$\left[12\frac{x^4}{4} - 21\frac{x^3}{3} + 10\frac{x^2}{2}\right]^k = \frac{1}{2}$$

$$\left[3x^4 - 7x^3 + 5x^2\right]_0^k = \frac{1}{2}$$

$$\left[3k^4 - 7k^3 + 5k^2 - 0\right] = \frac{1}{2}$$

$$\left[3k^4 - 7k^3 + 5k^2\right] = \frac{1}{2}$$

$$k^2\left(3x^2 - 7k + 5\right) = \frac{1}{2}$$

For k = 0.452; left hand side value will be equal to approximately $\frac{1}{2} = 0.5$

Example 1.111. X is a continuous random variable with probability density function given by **(JNTU 2003)**

$$f(x) = \begin{cases} kx & (0 \le x < 2) \\ 2k & (2 \le x < 4) \\ -kx + 6k & (4 \le x < 6) \end{cases}$$

Find k and mean value of X

Solution

We know that far the probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_0^2 f(x)\,dx + \int_2^4 f(x)dx + \int_4^6 f(x)dx + \int_6^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} o\,dx + \int_0^2 k\cdot dx + \int_2^4 2k\,dx + \int_4^6 (-kx + 6k)dx + \int_6^{\infty} 0\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^2 + 2k[x]_2^4 + \left[-k\frac{x^2}{2} + 6kx\right]_4^6 = 1$$

$$k\left[\frac{4}{2} - 0\right] + 2$$

$$2k + 4k - 18.$$

$$50k - 42k =$$

$$k = +\frac{1}{8}$$

Mean of $x =$

$$= \int_{-\infty}^{0} xf(x)dx$$

$$= \int_{-\infty}^{0} x.o\,dx +$$

$$= \int_0^2 k\,x^2 dx +$$

$$= k\left[\frac{x^3}{3}\right]_0^2 + 2.$$

$$= (k)\left(\frac{8}{3}\right) + (2.$$

$$= \frac{1}{8} \times \frac{8}{3} + \frac{2}{8} \times$$

$$= \frac{1}{3} + \frac{3}{2} - \frac{19}{3} +$$

$$= \frac{2 + 9 - 38 +}{6}$$

$$= \frac{56 - 38}{6} = \frac{18}{6}$$

Example 1.112. Is tl

$$F(x) = \begin{cases} 0 \\ \frac{k}{2a} \\ 1 \end{cases}$$

$$k\left[\frac{4}{2}-0\right]+2k[4-2]+\left[-k\frac{36}{2}+6k6+k\frac{16}{2}-6k4\right]=1$$

$$2k+4k-18k+36k+8k-24k=1$$

$$50k-42k=1,$$

$$k=+\frac{1}{8}$$

Mean of $x=\int_{-\infty}^{\infty} x\,f(x)\,dx$

$$=\int_{-\infty}^{0} xf(x)dx+\int_{0}^{2} x\,f(x)dx+\int_{2}^{4} x\,f(x)dx+\int_{4}^{6} x\;f(x)\;dx+\int_{6}^{\infty}\;x\;f(x)\;dx$$

$$=\int_{-\infty}^{0} x.o\,dx+\int_{0}^{2} x.(kx)dx+\int_{2}^{4} x.(2k)\;dx+\int_{4}^{6} x(-kx+6k)dx+\int_{6}^{8} x.o\;dx$$

$$=\int_{0}^{2} k\,x^2dx+\int_{2}^{4} 2kx\,dx+\int_{4}^{6}\left(-kx^2+6kx\right)dx$$

$$=k\left[\frac{x^3}{3}\right]_0^2+2k\left[\frac{x^2}{2}\right]_2^4+-k\left[\frac{x^3}{3}\right]_9^6+6k\left[\frac{x^2}{2}\right]_4^6$$

$$=(k)\left(\frac{8}{3}\right)+(2k)\left(\frac{16}{2}-\frac{4}{2}\right)-(k)\left(\frac{216}{3}-\frac{64}{3}\right)+(6k)\left(\frac{36}{2}-\frac{16}{2}\right)$$

$$=\frac{1}{8}\times\frac{8}{3}+\frac{2}{8}\times\frac{12}{2}-\frac{1}{8}\times\frac{152}{3}+\frac{6}{8}\times\frac{20}{2}$$

$$=\frac{1}{3}+\frac{3}{2}-\frac{19}{3}+\frac{15}{2}$$

$$=\frac{2+9-38+45}{6}$$

$$=\frac{56-38}{6}=\frac{18}{6}=3$$

Ans.

Example 1.112. Is the following function is a distribution function

$$F(x)=\begin{cases}0 & x<-a\\ \dfrac{k}{2a} & -a\le k\le a\\ 1 & x>a\end{cases}$$

Solution

We know that F′(x) = f(x)

$$\frac{d}{dx}[F(x)]=\frac{d}{dx}\left(\frac{x}{2a}\right)=\frac{1}{2a}=f(x)$$

$$f(x)=\begin{cases}0 & x<-a\\ \dfrac{1}{2a} & -a\le x\le a\\ 1 & x>a\end{cases}$$

If $\int_{-\infty}^{\infty} f(x)\,dx=1$ then f(x) is the probability density function

$$\int_{-\infty}^{-a} f(x)\,dx+\int_{-a}^{a} f(x)\,dx+\int_{a}^{\infty} f(x)\,dx=1$$

$$\int_{-\infty}^{-a} o\,dx+\int_{-a}^{a}\frac{1}{2a}\,dx+\int_{a}^{\infty} o\,dx=1$$

$$\frac{1}{2a}[x]_{-a}^{a}=1$$

$$\frac{1}{2a}[a-(-a)]=1$$

$$\frac{1}{2a}(2a)=1$$

$$1=1$$

Thus f(x) is a distribution function

Example 1.113. Is the following function is a distribution function

$$F(x)=\begin{cases}1-e^{-\frac{x}{2}} & x\ge 0\\ 0 & x<o\end{cases}$$

Solution

We know that F′(x) = f(x)

$$\frac{d}{dx}\left[-e^{-\frac{x}{2}}\right]=\frac{1}{2}e^{-\frac{x}{2}},\ f(x)=\begin{cases}\dfrac{1}{2}e^{\frac{-x}{2}} & x\ge 0\\ 0 & x<0\end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx=1$$

Thus the given

Example 1.114. Fi

variable x whose pro

Solution

Expected value = E

Variance σ^2

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} o\,dx + \int_{0}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = 1$$

$$\frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_{0}^{\infty} = 1$$

$$\left[-e^{-\frac{x}{2}}\right]_{0}^{\infty} = 1$$

$$\left[-e^{-\frac{\infty}{2}} + e^{0}\right] = 1$$

$$1 = 1$$

Thus the given function is a distribution function

Example 1.114. Find the expected value and variance of the discrete random variable x whose probability distribution is given below:

$X = x_i$	1	2	3	4	5	6
$P(X = x_i)$	0.25	0.21	0.51	0.01	0.01	0.01

Solution

Expected value = E(x)

$$= \sum_{i=1}^{6} x_i \, P(X = x_i)$$

$$= 1(2.25) + 2(0.21) + 3(0.51) + 4(0.01) + 5(0.01) + 6(0.01)$$

$$= 0.25 + 0.42 + 1.53 + 0.04 + 0.05 + 0.06$$

$$= 2.35$$

Ans.

Variance $\quad \sigma^2 = \sum_{i=1}^{6} (x_i - \mu)^2 \, P(X = x_i)$

$$= (1 - 2.35)^2\, 0.25 + (2 - 2.35)^2\, 0.21 + (3 - 2.35)^2\, 0.51 + (4 - 2.35)^2\, 0.01$$
$$+ (5 - 2.35)^2\, 0.01 + (6 - 2.35)^2\, 0.01$$

$$= (-1.35)^2(0.25) + (-0.35)^2(0.21) + (0.65)^2(0.51) + (1.65)^2(0.01)$$
$$+ (2.65)^2(0.01) + (3.65)^2(0.01)$$
$$= 1.8225 \times (0.25) + 0.1225 \times (0.21) + 0.4225(0.51) + 2.7225 \times (0.01)$$
$$+ 7.0225 \times (0.01) + 13.3225 \times (0.01)$$
$$= 0.4556 + 0.02572 + 0.21547 + 0.027225 + 0.070225 + 0.13325$$
$$= 0.92749$$ **Ans.**

Example 1.115. Given the following probability distribution

$X = x_i$	1	2	3	4	5
$P(X = x_i)$	0.1	0.3	0.2	0.2	0.2

Find

1. $E(X)$ 2. E (2 X +5)

3. $E\left(\dfrac{X}{2}+3\right)$ 4. $E(X^2)$

Solution

1. $E(X)$ $= \text{mean} = \sum_{1=1}^{5} x_i \ P(X = x_i)$

$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.2 + 5 \times 0.2$

$= 0.1 + 0.6 + 0.6 + 0.8 + 1$

$= 3.1$ **Ans.**

2. E (2X + 5) = E(2X) + E(5)

$= 2\,E(X) + 5$

$= 2 \times (3.1) + 5$

3. $E\left(\dfrac{X}{2}+3\right)$

4. $E(X)^2$

Example 1.116. A
= 3 and x = 6 has

Solution

We know that

$$= 6.2 + 5$$

$$= 11.2$$ **Ans.**

3. $E\left(\dfrac{X}{2}+3\right) = \dfrac{1}{2}E(X)+3$

$$= \frac{1}{2}\times(3.1)+3$$

$$= 1.55+3$$

$$= 4.55$$ **Ans.**

4. $E(X)^2 = \sum_{l=1}^{5} x_i^2 P(X = x_i)$

$$= 1^2\times(0.1)+2^2\times(0.3)+3^2(0.2)+4^2\times(0.2)+5^2\times(0.2)$$

$$= 1\times(0.1)+4\times(0.3)+9\times(0.2)+16\times(0.2)+25\times(0.2)$$

$$= 0.1+1.2+1.8+3.2+5$$

$$= 11.3$$ **Ans.**

Example 1.116. A continuous random variable X, can assume any value between x = 3 and x = 6 has a density function given by f(x) = k (1 + 2x) Find the value of k.

Solution

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_3^6 k(1+2x)dx = 1$$

$$k\left[k+\frac{2x^2}{2}\right]_3^6 = 1$$

$$k\left[x+x^2\right]_3^6 = 1$$

$$k[6-3+36-9]=1$$

$$k[42-9]=1$$

$$k[33]=1$$

$$k=\frac{1}{33} \qquad \textbf{Ans.}$$

Example 1.117. Find a constant a > 0 so that the given ~~function~~ is a density function.

$$f(x)=\begin{cases} \dfrac{e^{2x}}{2} & 0 \le x \le a \\ 0 & else\ where \end{cases}$$

Solution

We know that for a function to be a probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} o\,dx + \int_{0}^{a} \frac{e^{2x}}{2}\,dx + \int_{a}^{\infty} o\,dx = 1$$

$$\int_{0}^{a} \frac{c^{2x}}{2}\,dx = 1$$

$$\frac{1}{2}\left[\frac{e^{2x}}{2}\right]_0^a = 1$$

$$\frac{1}{4}\left[e^{2a}-3^{2.0}\right]=1$$

$$\frac{1}{4}\left[e^{2a}-1\right]=1$$

Taking log

Example 1.118. Th
deck. Determine the

Solution

Let X is a ran
three draws. In the dra

Clearly randon

The probability

$P(X=0)=\frac{36}{52}$

$P(X=1)=\frac{16}{52}$

$P(X=2)=\frac{16}{52}$

$$e^{2a} - 1 = 4$$

$$e^{2a} = 5$$

Taking log

$$2a = \log_e 5$$

$$2a = 1.609$$

$$a = 0.8047$$ **Ans.**

Example 1.118. Three cards are drawn without replacement from a well-shuttled deck. Determine the probability distribution of the number of face cards.

Solution

Let X is a random variable. Which is the number of face cards obtained in three draws. In the drawing of three cards number of face cards may be 0 or 1 or 2 or 3.

Clearly random variable X takes the values 0, 1, 2, 3

The probability of getting face cards $P = \frac{16}{52}$

$$q = 1 - \frac{16}{52}$$

$$= \frac{52-16}{52}$$

$$= \frac{36}{52}$$

$$P(X=0) = \frac{36}{52} \times \frac{35}{51} \times \frac{34}{50} = \frac{42840}{132600} = \frac{214}{663}$$

$$P(X=1) = \frac{16}{52} \times \frac{36}{51} \times \frac{35}{50} + \frac{36}{52} \times \frac{16}{51} \times \frac{35}{50} + \frac{36}{52} \times \frac{35}{51} \times \frac{16}{50} = \frac{1512}{3315}$$

$$P(X=2) = \frac{16}{52} \times \frac{15}{51} \times \frac{36}{50} + \frac{36}{52} \times \frac{16}{51} \times \frac{15}{50} + \frac{16}{60} \times \frac{36}{51} \times \frac{15}{50} = \frac{216}{3315}$$

$$P(X=3)=\frac{16}{52}\times\frac{15}{51}\times\frac{14}{50}=\frac{84}{3315}$$

Probability distribution

$X = x_i$	0	1	2	3
$P(X = x_i)$	$\frac{214}{663}$	$\frac{1512}{3315}$	$\frac{216}{3315}$	$\frac{84}{3315}$

Example 1.119. In the experiment of throwing of a pair of dice, find the probability distribution of the number of doublets in three throws.

Solution

Let X is the random variable that denotes the no. of doublets obtained in three throws of a pair of dice.

Clearly random variable X takes the values 0, 1, 2, 3

Probability of getting the number of doublets

$$P=\frac{6}{36}=\frac{1}{6}$$

$$q=1-\frac{1}{6}=\frac{5}{6}$$

P (of not getting number of doublets in three throws)

$$=P(x=0)=\frac{5}{6}\times\frac{6}{6}\times\frac{5}{6}=\frac{125}{216}$$

P (of getting doublets in any one throw and not getting doublets in remaining two throw)

$$=P(x=1)=\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{75}{216}$$

P (of getting doublets in any two throws and not getting doublets in remaining one throw)

$$= P(x=2) = \frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{15}{216}$$

P (of getting doublets in all three throws)

$$= \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{216}$$

Example 1.120. Find μ, σ^2 and σ for the given probability distribution

X_i	1	2	3	4
$P(X_i)$	0.5	0.2	0.2	0.1

Solution

X_i	P_i	P_iX_i	$X_i-\mu$	$(X_i-\mu)^2$	$(X_i - \mu)^2P_i$
1	**0.5**	**0.5**	**0.9**	**0.81**	**0.404**
2	**0.2**	**0.4**	**0.1**	**0.01**	**0.002**
3	**0.2**	**0.6**	**1.1**	**1.21**	**0.242**
4	**0.1**	**0.4**	**2.1**	**4.41**	**0.441**
		μ = 1.9			**σ^2 = 1.09**

$$Mean\ \mu = \sum P_i X_i$$

$$= 0.5+0.4+0.6+0.4$$

$$= 1.9$$

$$Variance\ \sigma^2 = \sum (X_i-\mu)^2 P_i$$

$$= 0.405+0.002+0.242+0.441$$

$$= 1.09$$

$$\sigma = \sqrt{1.09}$$

$$\sigma = 1.04$$

Example 1.121. Find the mean and the variance of the number of heads obtained, when 4 coins are through

Solution

In the experiment of tossing of 4 coins sample space is

S = {HHHH, HTHH, T HHH, HHTH, HHHT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, TTHH, THTH, THHT, TTTT}

Total numbers of possible cases are 16.

Number of heads may be 0 then probability is = 1/16

Number of heads may be 1 then probability is = 4/16

Number of heads may be 2 then probability is = 6/16

Number of heads may be 3 then probability is = 4/16

Number of heads may be 4 then probability is = 1/16

No. of heads $x = x_i$	$P(x = x_i)$	$p_i\, x_i$	$x_i - \mu$	$(x_i - \mu)^2$	$(x_I - \mu)^2\, p_i$
0	1/16	0	-2	4	4/16
1	4/16	4/16	-1	1	4/16
2	5/16	12/6	0	0	0
3	4/16	12/16	1	1	4/16
4	1/16	4/16	2	4	4/16

Mean = $\sum p_i x_i = 2$

Variance = $\sum (x_i - u)^2\ p_i = 1$

Probability

1) Determin

a) The su

b) No he

[Hint: a) su

Tota

b) P (N

2) Two balls
blue balls,

a) F

b) E

[Hint: Total r

a)

b)

3) Solve the al

[Hint: a) P (f

P

P

b) P (f

P(

P(

Conditional Prob

4) If A and B t

Find a) P (A

ads obtained,

THT, TTTH,

$(x_I - \mu)^2\ p_i$
4/16
4/16
0
4/16
4/16

EXERCISE

Probability

1) Determine the probability for each of the following:

   a) The sum 7 appears in a single toss of pair of fair dice.

   b) No head appears in a four tosses of a fair coin.

[**Hint**: a) sum 7 appears in 6 ways (6,1)(1,6)(5,2)(2,5)(4,3)(3,4)

Total no of cases =36 ways

P(sum 7)= Favourable cases/total cases = 6/36=1/6) **Ans.**

b) P (No head means all four tails) = P (T) P (T) P (T) P (T)

= 1/2×1/2×1/2×1/2=1/16 **Ans.**]

2) Two balls are drawn in succession from a box containing 5 red, 15 white, 10 blue balls, with replacement, one by one. Find the probability that

   a) First white and second red

   b) Both are red

[**Hint**: Total no of balls are 5+15+10=30

a) P(first white)=15/30

P(second red after replacement)=5/30

P(first white and second red)=15/30×5/30=1/12 **Ans.**

b) P(First red)=15/30

P(Second red after replacement)=5/30

P(Both are red)=5/30×5/30=1/36 **Ans.**]

3) Solve the above problem with no replacement

[**Hint**: a) P (first white) =15/30

P (second red with out replacement) =5/29

P (first white, second red) =15/30×5/29=75/870 **Ans.**

b) P (first white) =15/30

P(second white without replacement)=14/29

P(both white)=15/30×14/29=210/870 **Ans.**]

Conditional Probability

4) If A and B be events with P (A) =1/2, P (B) =1/4, P (AUB) =1/3

Find a) P (A/B), b) P (B/A)

[**Hint**: a) P (AUB) =P(A)+P(B)-P(A∩B)

1/3 =1/2 + 1/4 –P (A∩B)

P (A∩B) =5/12

P (A/B) = P (A∩B)/P (B) =5/12/1/4=5/3 **Ans.**

P (B/A) =P (A∩B)/P (A) =5/12/1/2=5/6 **Ans.**]

5) A college has 10 male teachers and 5 female teachers.3 teachers are selected at random one by one. Find probability that

a) First two are male teachers and third is female teacher.

b) First and third are male teacher and second is female teacher.

[**Hint**:(a) P(M1 and M2 and F3)= P(M1 M2 F3)=P(M1)P(M2/M1)P(F3/M1M2)

= 10/15× 9/14 × 5/13

= 15/91 **Ans.**

(b) P(M1and F2 and M3)= P(M1) P(F2/M1) P(M3/M1F2)

= 10/15 × 5/14 × 9/13

= 15/91 **Ans.**]

Theorem on Total Probability

6) Three machines M_A, M_B, M_C produce respectively 20%, 30%, 50% of the total number of items. Out of these 5%, 4% and 3% items are defective. If an item is selected at random, find the probability that the item is defective.

[Hint:P(defective) = $P(M_A)P(D/M_A)+ P(M_B)P(D/M_B)+ P(M_C)P(D/M_C)$

= (0.2)(0.05) + (0.3)(0.04) + (0.5)(0.03)

= 0.037 **Ans.**]

Bayes theorem

7) Three machines I,II and III produce 40%, 30% and 30% of the total no of items of a factory. The percentage of defective items of these machines is 4%, 2% and 3%. If an item is selected at random and to be defective, find the probability that

1) it is from machine I

2) it is from machine II

3) it is from machine III **(Supple.Feb.2010 Set 4)**

[Hint: P(

P(M

P(M

8) Compani
wagon R
Company

a) What i

b) What i

co

[Hint: (a) P(

(b) P

Probability Dist

9) determine

x_i 1

f_i 0

[Hint: μ=1×0.3

σ^2= (1-2.7)^2× (

= 0.867+1.0

= 2.01

σ = 1.4177 An

10) A bag conta
from bag. D

[Hint: X

P(x_i)

0×7/1

[**Hint**: $P(M_I/D)=P(M_I)P(D/M_I)/[P(M_I)P(M_I)+ P(M_{II}) P(D/M_{II})+ P(M_{III}) P(D/M_{III})]$

$= (0.3)(0.02)/[(0.4)(0.04)+(0.3)(0.02)+(0.3)(0.03)]$

$= 0.016/0.00909$ **Ans.**

$P(M_{II}/D)=(0.3)(0.02)/[(0.4)(0.04)+(0.3)(0.02)+(0.3)(0.03)$

$= 0.006/0.00909$ **Ans.**

$P(M_{III}/D)= (0.3)(0.03)/(0.4)(0.04)+(0.3)(0.02)+(0.3)(0.03)$

$= 0.009/0.00909$ **Ans.**]

8) Companies A, B, C produce 30%, 45% and 25% of maruthi800, Alto and wagon R respectively. 2% of maruthi800 from company A, 3% of Alto from Company B and 2% of wagon R from Company C are defective.

a) What is the probability that a car purchased is defective?

b) What is the probability that this car is maruthi800 and it is produced by company A.?

[**Hint**: (a) P(Defective)=(0.3)(0.02)+(0.45)(0.03)+(0.25)(0.02)

=0.0245 **Ans.**

(b) P (A/D) = (0.3)(0.02) / 0.0245

= 0.006/0.0245 **Ans.**]

Probability Distribution:

9) determine μ, σ^2,σ for

x_i	1	5	2	3
f_i	0.3	0.2	0.1	0.4

[**Hint**: $\mu=1\times0.3+5\times0.2+2\times0.1+3\times0.4$

$\sigma^{2}= (1\text{-}2.7)^2\times (0.3) +(5\text{-}2.7)^2\times (0.2)+(2\text{-}2.7)^2\times (0.1)+(3\text{-}2.7)^2\times0.4$

$= 0.867+1.058+0.049+0.036$

$= 2.01$

$\sigma = 1.4177$ **Ans.**]

10) A bag contains 10 bulbs of which 2 are defective. Three bulbs are taken out from bag. Determine the expected no of defective bulbs which are taken out.

[**Hint**:

X	0	1	2
$P(x_i)$	$8c_3 2c_0/10c_3$	$8c_2 2c_1/10c_3$	$8c_1 2c_2/10c_3$
	7/15	7/15	1/15

$0\times7/15 + 1\times7/15 + 2\times1/15 = 3/5$ **Ans.**]

OBJECTIVE TYPE QUESTIONS

1. The probability of getting a number greater than 2 or an even number in a single throw of a fair die is

   (a) 1/3 (b) 1/2 (c) 5/6 (d) none of these

Ans. (a)

2. A single die is tossed once. The probability of a 2 or 5 turning up.

   (a) 1/3 (b) 2/3 (c) 5/6 (d) none of these

Ans. (a)

3. If A and B are any two arbitrary events of S then

   (a) $P(A \cup B) = P(A) + P(B)$

   (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   (c) $P(A \cup B) = P(A) + P(B) - P(A/B)$ (d) none of these

Ans. (b)

4. Two events A and B are independent events if and only if

   (a) $P(A \cup B) = P(A) + P(B)$ (b) $P(A \cap B) = P(A) + P(B)$

   (c) $P(A \cap B) = P(A).P(B)$ (d) none of these

Ans. (c)

5. In the experiment of throwing a pair of dice, The probability of getting two equal number.

   (a) 1/3 (b) 1/6 (c) 5/6 (d) none of these

Ans. (b)

6. By the conditional probability which is correct

   (a) $P(A/B) = \dfrac{P(A \cup B)}{P(B)}$ (b) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

(c) $P(A/B)$

Ans. (b)

7. What is the

   (a) 4/52

Ans. (c)

8. From 20 tick

   probability t

   (a) 13/20

Ans. (c)

9. If a card is d
   that it is a sp

   (a) 4/52

Ans. (d)

10. A book conta

    chance that th

    (a) 1/10

Ans. (a)

11. Under which
    become a pr

    (a) $P(a_1) = 0$

    (b) $P(a_1) = ($

    (c) $P(a_1) = ($

    (d) $P(a_1) = 0$

Ans. (c)

12. Suppose A an

    $P(A \cap B) = 0.4$.

(c) $P(A/B)=\dfrac{P(B)}{P(A\cap B)}$ (d) none of these

Ans. (b)

7. What is the probability of choosing any 4 or 7 in the deck of 52 cards?

   (a) 4/52 (b) 2/26 (c) 8/52 (d) none of these

Ans. (c)

8. From 20 tickets marked 1, 2, 3,........, 20 one is drawn at random. The probability that the number on it is a multiple of 2 or 3 is.

   (a) 13/20 (b) 3/20 (c) 16/20 (d) none of these

Ans. (c)

9. If a card is drawn from a well shuffled pack of 52 cards , then the probability that it is a spade or a queen is

   (a) 4/52 (b) 4/13 (c) 17/52 (d) none of these

Ans. (d)

10. A book contains 100 pages. A page chosen at random. What is the chance that the sum of digits on a page is equal to 9.

    (a) 1/10 (b) 15/100 (c) 17/100 (d) none of these

Ans. (a)

11. Under which of the following functions does $S=\{a_1, a_2, a_3\}$ become a probability space?

    (a) $P(a_1) = 0.3$, $P(a_2) = 0.4$, $P(a_3) = 0.5$
    (b) $P(a_1) = 0.8$, $P(a_2) = -0.2$, $P(a_3) = 0.4$
    (c) $P(a_1) = 0.2$, $P(a_2) = 0.3$, $P(a_3) = 0.5$
    (d) $P(a_1) = 0.3$, $P(a_2) = 0.1$, $P(a_3) = 0.7$

Ans. (c)

12. Suppose A and B are events with P(A)= 0.7, P(B)= 0.5, and $P(A\cap B)= 0.4$ .What will be the probability that A does not occur

(a) 0.7 (b) 0.3 (c) 0.9 (d) None of the above

Ans. (b)

13. In the following probability distribution does S = {1, 2, 3, 4, 5, 6} become a probability space?

Outcome	1	2	3	4	5	6
Probability	0.1	0.3	0.2	0.2	0.1	0.1

(a) Yes (b) No

(c) Some times it may be probability space (d) None of the above

Ans. (a)

14. If Sample Space is $S = \{a_1, a_2, a_3\}$ and $P(a_2)=0.3, P(a_3)=0.5$. Then $P(a_1)$ is

(a) 0.3 (b) 0.5 (c) 0.2 (d) None of the above

Ans. (c)

15. A digit is selected at random from the digit 1 to 9. If the event A= {1,3,5,7,9} and B={2,3,5,7} then $P(A \cap B)$ is

(a) $\frac{7}{9}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) None of the above

Ans. (b)

16. In reference of above question if C = {6, 7, 8, 9} then $P(A \cap C)$ is

(a) $\frac{3}{9}$ (b) $\frac{7}{9}$ (c) $\frac{5}{9}$ (d) None of the above

17. In the experiment of tossing of the three coins number of heads is the random variable then the probability of 0 heads is Random variable X = number of heads

(a) 2/8 (b) 3/8 (c) 1/8 (d) none of these

Ans. (c)

Hint. Sample Space = {HHH, HTT, HTH, THH, THT, HHT, TTH, TTT}

18. For the discret

(a) $\sum_{all\,x} f(x)=$

(c) $\sum_{all\,x} f(x)\neq$

Ans. (a)

19. for the discret

(a) Mean $= \sum$

(b)Mean $= \sum$

(c)Mean $= \sum$

(d)None of th

Ans. (a)

20. If a ball is drav find the probab

(a) 2/5

Ans. (a)

Hint. $P(x<5)=\sum_{x=0}^{4}$

21. The probability Prime

(a) 4/20

Ans. (b)

Hint. Prime numbers

Probability of pı

$= \frac{1}{20}+\frac{1}{20}+$

18. For the discrete probability distribution

(a) $\sum_{all\,x} f(x) = \sum_{all\,x} P(X = x) = 1$ (b) $\sum_{all\,x} f(x) = \sum_{all\,x} P(X = x) = 1/2$

(c) $\sum_{all\,x} f(x) \neq \sum_{all\,x} P(X = x) = 1$ (d) none of these

Ans. (a)

19. for the discrete probability distribution

(a) Mean $= \sum x\ f(x)$, and Variance $\sigma^2 = \sum_i (X_i - \mu)^2 f(x_i)$

(b)Mean $= \sum\ f(x)$, and Variance $\sigma^2 = \sum_i (X_i - \mu)^2 f(x_i)$

(c)Mean $= \sum x\ f(x)$, and Variance $\sigma^2 = \sum_i (X_i - \mu) f(x_i)$

(d)None of these

Ans. (a)

20. If a ball is drawn from a box containing 10 balls numbered 1 to 10 inclusive find the probability that the number X drawn is less than 5

(a) 2/5   (b) 3/10   (c) 5/10   (d) none of these

Ans. (a)

Hint. $P(x<5) = \sum_{x=0}^{4} P(x) = \sum_{x=0}^{4} \frac{1}{10} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$

21. The probability that a card drawn at random from 20 cards numbered 1 to 20 Prime

(a) 4/20   (b) 8/20   (c) 2/10   (d) none of these

Ans. (b)

Hint. Prime numbers are (2, 3, 5, 7, 11, 13, 17, 19) with the probability 1/20 each.

Probability of prime number

$$= \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} = \frac{8}{20}$$

22. Maximum value of probability is

(a) 0 (b) 1 (c) ½ (d) none of these

Ans. (b)

23. The mean of the probability distribution of the number of heads obtained in two flips of a balanced coin is a .

(a) 1 (b) ½ (c) 0 (d) none of these

Ans. (a)

24. A random variable X has the density function $f(x) = c/(x^2+1)$, where $-\infty < x < +\infty$. The value of the constant c

(a) $1/\pi$ (b) $2/\pi$ (c) $3/\pi$ (d) none of these

Ans. (a)

Hint. $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{+\infty} \frac{c}{x^2+1} dx = c \tan^{-1} x \Big|_{-\infty}^{+\infty} = c\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$

25. The value of the constant c such that the function is a density function.

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & otherwise \end{cases}$$

(a) 2/9 (b) 1/3 (c) 1/9 (d) none of these

Ans. (c)

Hint. $\int_{-\infty}^{+\infty} f(x)dx = \int_0^3 cx^2 dx = \frac{cx^3}{3}\Big|_0^3 = 9c$

26. The value of the constant c = 1/9 such that the function is a density function then P (1< x <2).

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & otherwise \end{cases}$$

(a) 2/9 (b) 5/27 (c) 7/27 (d) none of these

Ans. (c)

Hint. $P(1< x <2) = \int_1^2 \frac{1}{9} x^2 dx = \frac{x^3}{27}\Big|_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$

27. A fair coin is t
S = { HHH, HI
variable which
space is
(a) $R_x = \{0, 1,$
(c) $R_x = \{0, 1\}$
Ans. (b)

28. Let X be rando

X_i
P_i

Then E(X) will
(a) $E(X) = \sum X_i$
(c) $E(X) = \sum P_i$
Ans. (a)

29. In reference of
(a) 5
Ans. (b)
[Hint: $E(X) = \sum X_i P_i =$

30. Let X be a rand
1) E (KX) = K E
Thus for any rea
(a) E(aX + b) =
(c) E (aX + b) =
Ans. (b)

31. Suppose a fair di
$x_3 = 4$, $x_4 = 2$, x_5
(a) 3.27
Ans. (b)

[Hint: Sample M

27. A fair coin is tossed three times, with the Sample Space

S = { HHH, HHT, HTH, HTT, THH, THT, TTH, TTT } Let X be the random variable which assigns to each point in S the number of heads. Then the range space is

(a) $R_x = \{0, 1, 2\}$ (b) $R_x = \{0, 1, 2, 3\}$

(c) $R_x = \{0, 1\}$ (d) none of these

Ans. (b)

28. Let X be random variables with the following respective distributions

X_i	2	3	6	10
P_i	0.2	0.1	0.4	0.3

Then E(X) will be given by

(a) $E(X) = \sum X_i P_i$ (b) $E(X) = \sum X_i$

(c) $E(X) = \sum P_i$ (d) None of the above

Ans. (a)

29. In reference of above question E(X) is

(a) 5 (b) 6.1 (c) 6 (d) None of the above

Ans. (b)

[**Hint:** $E(X) = \sum X_i P_i = 2 \times 0.2 + 3 \times 0.1 + 6 \times 0.4 + 10 \times 0.3 = 6.1$]

30. Let X be a random variable and let K be a real number. Then

1) E (KX) = K E(X) and 2) E(X + K) = E(X) + K

Thus for any real numbers a and b

(a) E(aX + b) = a + b E(X) (b) E(aX + b) = aE(X) +b

(c) E (aX + b) = b + E(X) (d) None of the above

Ans. (b)

31. Suppose a fair die is tossed 8 times with the following outcomes $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 2$, $x_5 = 1$, $x_6 = 3$, $x_7 = 5$, $x_8 = 6$, then the Sample mean is

(a) 3.27 (b) 3.25 (c) 3.3 (d) None of the above

Ans. (b)

[**Hint:** Sample Mean $= \dfrac{2+3+4+2+1+3+5+6}{8} = \dfrac{26}{8} = 3.25$]

32. In the following Probability distributions what is the value of $E(X^2)$ and variance σ^2 respectively

X_i	-5	-4	1	2
P_i	1/4	1/8	1/2	1/8

(a) $E(X^2) = 9.25$ $\quad \sigma^2 = 8$ $\quad$ (b) $E(X^2) = 9.25$ $\quad \sigma^2 = 7.25$

(c) $E(X^2) = 9.25$ $\quad \sigma^2 = 8.25$ $\quad$ (d) None of the above

Ans. (c)

[Hint: $E(X^2) = \sum X_i^2 P_i = 25(1/4) + 16(1/8) + 1(1/2) + 4(1/8) = 9.25$

$\sigma^2 = E(X^2) - \mu^2 = 9.25 - (-1)^2 = 8.25$]

7.25

UNIT-2

DISTRIBUTIONS

"Nothing will work unless you do."

2.1 BERNOULLI PROCESS

1. There are n trials where n is a fixed integer.
2. The out comes in all n trials are independent
3. For each trial there are only two possible outcomes arbitrarily called success and failure.
4. The probability of the success denoted by P and it is same for each trial.

Binomial distribution is a discrete probability distribution for the binomial random variable X, where X is the number of success and numbers of Bernoulli trials are n then binomial distribution is given by

$$b(x;n,p)=n_{c_x}\ p^x\ q^{n-x},\quad x=0,1,2\ldots\ldots n$$

Where P is the probability of the happening the event i.e., the probability of the success and q is the probability of the failure i.e. probability of the not happening the event.

$$(q+p)^n=q^n+n_{c_1}\ pq^{n-1}+n_{c_2}\ p^2q^{n-2}+\ldots\ldots+p^n$$

By the binomial expansion

$$= n_{c_0} q^n + n_{c_1} pq^{n-1} + n_{c_2} p^2 q^{n-2} + \ldots\ldots + n_{c_n} p^n$$

$$= b(0;n,p) + b(1;n,p) + b(2;n,p) \ldots\ldots + b(n;n,p)$$

$$= \sum_{x=0}^{n} b(x;n,p)$$

We know that q + p = 1

$$\Rightarrow (q+p)^n = \sum_{x=0}^{n} b(x;n,p) = 1$$

2.2. RECURRENCE RELATION FOR BINOMIAL DISTRIBUTION

$$b(x;n,p) = n_{c_x} p^x q^{n-x}$$

$$b(x;n,p) = \frac{\angle n}{\angle x \ \angle n-x} \ p^x \ q^{n-x}$$

$$b(x+1;n,p) = \frac{\angle n}{\angle x+1 \ \angle n-x-1} p^{x+1} q^{n-x-1}$$

$$= \frac{(n-x) \ \angle n}{(x+1) \ \angle x \ (n-x) \ \angle n-x-1} \ p \ p^x q^{-1} \ q^{n-x}$$

$$= \frac{(n-x)}{(x+1)} \ \frac{p}{q} \ \frac{\angle n}{\angle x \angle n-x} \ p^x q^{n-x}$$

$$b(x+1;\ n,\ p) = \frac{n-x}{x+1} \ \frac{p}{q} \ b(x;\ n,\ p)$$

Which is the Recurrence Relation.

2.3 CONSTANTS OF THE BINOMIAL DISTRIBUTION

When one can say a distribution is binomial?

Find the constants of the binomial distribution.

(Reg. April/May 2004 Set 1)

Mean of the binomial distribution

We know that

$$\text{Mean} = \sum_{x=0}^{n} x \ p(x)$$

Let x – 1 = y v

Again let n – 1

Variance of the bino

Variance $\sigma^2 = \sum_{x=0}^{n} (x$

Variance $\sigma^2 = E(X)$

$$= \sum_{x=0}^{n} x \; b(x;\, n,\, p)$$

$$= \sum_{x=0}^{n} x.\, n_{c_x}\, p^{x}\, q^{n-x}$$

$$= \sum_{x=0}^{n} x \;\frac{\angle n}{\angle x \angle n-x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{x.n \angle n-1}{x \angle x-1 \angle n-x}\; p\, p^{x-1} q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{np \angle n-1}{\angle x-1 \angle n-x}\; p^{x-1} q^{n-x}$$

Let x – 1 = y when x = 1, y = 0, x = n, y = n-1

$$= \sum_{y=0}^{n-1} \frac{np \angle n-1}{\angle y \angle n-1-y}\; p^{y} q^{n-1-y}$$

Again let n – 1 = m

$$= \sum_{y=0}^{m} np \;\frac{\angle m}{\angle y \angle m-y}\; p^{y}\; q^{m-y}$$

$$= \sum_{y=0}^{m} n\, p\, b(y : m,\, p)$$

$$= np \sum_{y=o}^{m} b(y;\, m,\, p)$$

$$\boxed{\text{Mean} = \text{n p}}$$

Variance of the binomial distribution

Variance $\sigma^2 = \sum_{x=0}^{n} (x-\mu)^2\, p(x)$

Variance $\sigma^2 = E(X)^2 - [E(X)]^2$ ....(1)

Consider

$$E(X)^2 = \sum_{x=0}^{n} x^2 P(X = x)$$

$$= \sum_{x=0}^{n} [x(x-1) + x] p(x)$$

$$= \sum_{x=0}^{n} x(x-1) p(x) + \sum_{x=0}^{n} x \; p(x)$$

$$= \sum_{x=0}^{n} x(x-1) \, b(x : n, p) + n \, p$$

$$= \sum_{x=0}^{n} \frac{x(x-1) \;\; \angle n \;\; p^x q^{n-x}}{\angle x \; \angle n-x} + n \; p$$

$$= \sum_{x=0}^{n} \; \frac{x(x-1) \;\; \angle n}{x(x-1) \; \angle x-2 \; \angle n-x} \; p^x \; q^{n-x} + n \; p$$

$$= \sum_{x=2}^{n} \frac{n(n-1)p^2 \angle n-2}{\angle x-2 \;\; \angle n-x} \; p^{x-2} q^{n-x} + n \; p$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{\angle n-2}{\angle x-2 \;\; \angle n-x} p^{x-2} q^{n-x} + n \; p$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \; \frac{\angle n-2}{\angle \mathbf{x}-2 \;\; \angle n-2-(n-2)} \; p^{x-2} q^{n-2-(x-2)} + n \; p$$

$$= n(n-1)p^2 (p+q)^{n-2} + n \; p$$

$$= n(n-1)p^2 + np \quad [\text{p} + \text{q} = 1]$$

From the equation (1)

$$\sigma^2 = n(n-1)p^2 + np - (np)^2 = n^2 p^2 - np^2 + np - n^2 p^2$$

$$= np(1-p) = n \, p \, q$$

2.4 POISSON DISTRIBUTION ITS VARIANCE AND MEAN

Define Poisson distribution and find its variance and mean.

(Reg. Nov 2006 Set 2, Set 3 Set 4)

OR

When one
from binom
Prove that

$\lim_{\substack{n \to \infty \\ p \to 0}} b($

For $\lambda =$ n. p

Poisson di

When n is
large and t
product n p

The Poisso

Poisson dis

Consider

=

=

=

$b(x; n, p) =$

$\lim_{n \to \infty} b(x : n, p$

When one can say a distribution is Poisson? Deduce the Poisson distribution from binomial distribution and find its constants.

Prove that

$$\lim_{\substack{n\to\infty \\ p\to 0}} b(x;\ n, p) = f(x;\ \lambda)$$

For $\lambda = n.\ p =$ constant

Poisson distribution **(Nov. 2009 set 4)**

When n is very large and P is very small, means number of trials are infinitely large and the probability of success P is very small ($n\to\infty$, $p\to\infty$) and the product $n\ p = \lambda$ (constant)

The Poisson distribution, with mean λ ($\lambda > 0$) given by

$$f(x;\ \lambda) = \frac{\lambda^x e^{-\lambda}}{\angle x} \quad \text{For } x = 0, 1, 2\ldots$$

Poisson distribution is also a discrete probability distribution

$$\text{Consider } b(x;\ n, p) = n_{c_x}\, p^x q^{n-x}$$

$$= \frac{\angle n}{\angle x \angle n-x}\, p^x q^{n-x}$$

$$= \frac{\angle n}{\angle x\ \angle n-x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \qquad \left[np = \lambda \Rightarrow p = \frac{\lambda}{n}\right]$$

$$= \frac{n(n-1)(n-2)..(n-(x-1))}{\angle x\ n^x}(\lambda)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{x-1}{n}\right)}{\angle x}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$b(x;n,p) = \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)..\left(1-\frac{x-1}{n}\right)\lambda^x\left(1-\frac{\lambda}{n}\right)^n}{\angle x\ \left(1-\frac{\lambda}{n}\right)^x}$$

$$\lim_{n\to\infty} b(x:n,p) = \frac{1.1\ldots\ldots\ldots\ldots..1\quad \lambda^x.e^{-\lambda}}{\angle x,\quad 1}$$

$$\lim_{n\to\infty} b(x;\, n,\, p) = \frac{\lambda^x e^{-\lambda}}{\angle x}$$

$$\left[\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x} = \left[\left(1-\frac{\lambda}{n}\right)^{\frac{n}{\lambda}}\right]^{\lambda}\left(1-\frac{\lambda}{n}\right)^{x} = e^{-\lambda}.1 = e^{-\lambda}\right]$$

$$\lim_{\substack{n\to\infty\\ p\to 0}} b(x;\, n,\, p) = f(x;\, \lambda)$$

Mean of the Poisson distribution **(Nov.2009 set 4)**

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} x\ p(X = x)$$

$$= \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^x}{\angle x}$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x\ \angle x-1}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{\angle x-1}$$

$$= e^{-\lambda}\sum_{x=1}^{\infty} \frac{\lambda\ \lambda^{x-1}}{\angle x-1}$$

$$= \lambda e^{-\lambda}\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{\angle x-1}$$

$$= \lambda e^{-\lambda}\left[\frac{\lambda^{1-1}}{\angle 1-1} + \frac{\lambda^{2-1}}{\angle 2-1} + \frac{\lambda^{3-1}}{\angle 3-1} + \ldots\ldots\ldots\right]$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{\angle 1} + \frac{\lambda^2}{\angle 2} + \ldots\ldots\right]$$

$$= \lambda e^{-\lambda}.e^{\lambda}$$

$$\left[e^x = 1 + \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \ldots\ldots\ldots\right]$$

$$\boxed{\text{Mean} = E(X) = \lambda}$$

Variance of th

Variance σ^2 =

σ^2 =

Consider

$E(X^2)$

=

=

=

=

=

=

=

=

=

=

From the equa

Variance σ^2

σ

Variance of the Poisson distribution **(Nov.2009 set 4)**

Variance $\sigma^2 = E(X^2) - [E(X)]^2$

$$\sigma^2 = E(X^2) - \lambda^2 \text{.............}(1)$$

Consider

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \; f(x, \lambda)$$

$$= \sum_{x=0}^{\infty} x^2 . \frac{e^{-\lambda} \; \lambda^x}{\angle x}$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] \frac{e^{-\lambda} \lambda^x}{\angle x}$$

$$= \sum_{x=0}^{\infty} \frac{x(x-1) e^{-\lambda} \lambda^x}{\angle x} + \sum_{x=0}^{\infty} \frac{x \; e^{-\lambda} \lambda^x}{\angle x}$$

$$= \sum_{x=0}^{\infty} \frac{x(x-1) \; e^{-\lambda} \lambda^x}{x(x-1) \; \angle x-2} + \lambda$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^x}{\angle x-2} + \lambda$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^2 \lambda^{\,x-2}}{\angle x-2} + \lambda$$

$$= e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{\angle x-2} + \lambda$$

$$= e^{-\lambda} \lambda^2 \left[\frac{\lambda^0}{\angle 0} + \frac{\lambda}{\angle 1} + \frac{\lambda^2}{\angle 2} + \text{..............} \right] + \lambda$$

$$= e^{-\lambda} \lambda^2 e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

From the equation (1)

Variance $\sigma^2 = \lambda^2 + \lambda - \lambda^2$

$$\boxed{\sigma^2 = \lambda}$$

2.5 NORMAL DISTRIBUTION

Write all the characteristics of the Normal distribution?

(Reg. April/May 2004 Set 4)
(Nov.2009 set 2)

Normal Distribution

Normal distribution is the probability distribution of a continuous random variable X i.e. the normal distribution is the continuous probability distribution.

It is given by

$$N(\mu,\ \sigma) = f(X) = \frac{1}{\sigma\sqrt{2x}} \exp\left[-\frac{(x-\mu)^2}{2\ \sigma^2}\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{(x-\mu)^2}{2\ \sigma^2}} \qquad [-\infty < x < \infty, \sigma > 0]$$

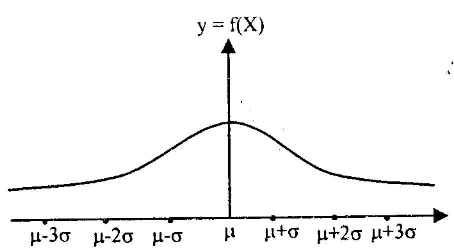
Where μ is mean, σ^2 is variance.

The graph of the normal distribution y = f(x) in the XY – plane is known as normal curve. Area under the normal curve is unity. Normal curve is symmetrical about y-areas and it is bell shaped. Probability between to continuous random variable X_1 and X_2 is given by

$$P(X_1 \leq X \leq X_2) = \int_{X_1}^{X_2} \frac{1}{\sigma\sqrt{2x}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Standard variable is given by

$$Z = \frac{x-\mu}{\sigma}$$



Normal Distribution Curve

Probability between two continuous random variable X_1, and X_2

$P(X_1 \leq X \leq X_2) = P(Z_1 \leq Z \leq Z_2)$

$= P(Z_2) - P(Z_1)$

= (Area under the normal curve from 0 to Z_2)

- (Area under the normal curve from 0 to Z_1)

2.6 VARIANCE, MODE, MEDIAN OF THE NORMAL DISTRIBUTION

(a) Prove that the mean = mode = median for a normal distribution

(Supple. Nov/Dec 2004)
(Supple. Feb. 2007 Set 3)

(b) Find the arithmetic mean of the normal distribution

(Supple. Nov/Dec 2004)
(Supple. Nov 2008.Set.2)

Solution

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$$

$$= \int_{-\infty}^{\infty} x.\frac{1}{\sigma\sqrt{2\,\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$

Let $\dfrac{x-\mu}{\sigma} = Z$

X - μ = Z σ

dx = σ dz

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} (\sigma z + \mu)\left(\frac{1}{\sigma\sqrt{2\pi}}\right) e^{-\frac{Z^2}{2}}\sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\; z + \mu) e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma\, z\, e^{-\frac{z^2}{2}}\, dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu\, e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}} dz + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}}\,(0) + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}}\,(0) + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, dz$$

$$= 0 + \mu(1)$$

$$\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, dz = 1\right]$$

Mean = E(X) = μ

Variance of the Normal Distribution

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 \; f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \; \frac{1}{\sigma\sqrt{2\pi}} . e^{\frac{-(x-\mu)^2}{2\sigma 2}} \, dx$$

Let $\dfrac{x-\mu}{\sigma} = z$

$x - \mu = \sigma z$

$dx = \sigma \, dz$

$$\text{Variance} = \int_{-\infty}^{\infty} (\sigma z)^2 \; \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{z^2}{2}} \; \sigma \, d z$$

$$= \int_{-\infty}^{\infty} \sigma^2 z^2 \; \frac{1}{\sqrt{2\pi}} \; e^{-\frac{z^2}{2}} \; dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \; e^{-\frac{z^2}{2}} \; dz = -\frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \; d\left(e^{-\frac{z^2}{2}}\right) dz$$

Integrating by parts

$$= -\frac{\sigma^2}{\sqrt{2\pi}} \left\{ \left[-ze^{-\frac{z^2}{2}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz \right\}$$

$$= -\frac{\sigma^2}{\sqrt{2\pi}} (0) + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} (0) + \sigma^2 \; (1)$$

Variance = σ^2

Mode of the Normal Distribution

We know that mode is the value of x for which function value is maximum i.e. f(x) = N (μ, σ) is maximum i.e. $f'(x) = 0$ and $f''(x) = -ve$

$$N(\mu,\sigma) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \; \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right]$$

i.e. x =

$\dfrac{d^2}{dx^2} [f(x)] = f$

= -ve (whe

$\Rightarrow$ x = μ is

Median of the nc

Suppose m is the

$\int_{-\infty}^{m} f(x) \, d$

We consider

$\int_{-\infty}^{\mu} \dfrac{1}{\sigma\sqrt{2\pi}}$

Now we con

By supposing

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{d}{dx}[f(x)] = f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(-\frac{(x-\mu)}{\sigma^2}\right)$$

Since $e^{-\frac{(x-\mu)^2}{2\sigma^2}} \neq 0$

i.e. x = μ, to be $f'(x) = 0$ ....(1)

$$\frac{d^2}{dx^2}[f(x)] = f''(x) = -\frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left(+\frac{1}{\sigma^2}\right) + \frac{(x-\mu)}{\sigma^2} . e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left(-\frac{(x-\mu)}{\sigma^2}\right)\right]$$

= -ve (when x = μ .... (2)

$\Rightarrow$ x = μ is the mode of the normal distribution

Median of the normal distribution

Suppose m is the median of the normal distribution then

$$\int_{-\infty}^{m} f(x)\, dx = \frac{1}{2} \quad \text{Or} \quad \int_{m}^{\infty} f(x)\, dx = \frac{1}{2}$$

We consider $\int_{-\infty}^{m} f(x)\, dx = \frac{1}{2}$

$$\int_{-\infty}^{\mu} f(x)\, dx + \int_{\mu}^{m} f(x)\, dx = \frac{1}{2}$$

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{\mu}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{2} \text{.........}(A)$$

Now we consider only first term

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx =$$

$$\int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma\, dz \text{...........}(B)$$

By supposing $\frac{x-\mu}{\sigma} = z$

When $x = -\infty$, $z = -\infty$

$x = \mu$, $z = 0$

From equation (A)

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\lambda}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = \frac{1}{2}$$

From Equation .... (A)

$$\frac{1}{2} + \int_{\mu}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{2}$$

$$\Rightarrow \int_{\mu}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 0$$

$\Rightarrow \mu = m$ = median of the normal distribution

2.7 POPULATION

Population is the set or collection of objects, under study.

2.8 SIZE

The number of objects or observation in the population is the size of population and denoted by N. Size N being finite or infinite

2.9 SAMPLE

A finite subset of the population known as sample.

OR

A small section selected from the population is called a sample size of the sample is denoted by n.

2.10 SAMPLING

The process of drawing a sample is called sampling.

Population	Sample
Engineering graduate students in A.P.	Engineering Graduate student of a college
Total production of items in a month	Total production of items in one day

2.10.1 Large Sam

2.10.2 Small Sam

2.10.3 Random S member of included in by this metl

2.10.4 Sampling the populat of the popu

2.10.5 Sampling the populat member of

2.11 PARAMETE

Statistical population paramet

Examples –

Pop

Pop

Pop

2.12 STATISTIC

Statistical co statistics or statistics

Examples –

Sam

Sam

Sam

Note: Paramet

2.13 SAMPLING D

Consider a random. For each sa not be identical. If the frequency distrib similarly we can hav

While drawing parent population re

2.10.1 Large Sampling: The sampling is said to be large sampling if $n \geq 30$

2.10.2 Small Sampling: The sampling is said to be small sampling if $n < 30$

2.10.3 Random Sampling: Random sampling is the sampling in which each member of the population has equal chances or probability of being included in the sample is know as random sampling. The sample obtained by this method is known as a random sample.

2.10.4 Sampling with replacement: After sampling each member is replaced in the population. That mean in the sampling with replacement each member of the population may be chosen more than once

2.10.5 Sampling without replacement: After sampling, element is not replaced in the population. That means in the sampling without replacement each member of the population can be chosen only once.

2.11 PARAMETERS

Statistical constants obtained from the population are known as population parameters or parameters.

Examples –

Population mean - μ

Population standard deviation - σ

Population proportion – p (small)

2.12 STATISTICS

Statistical constants obtained from sample observations are known as sample statistics or statistics.

Examples –

Sample mean - $\overline{x}$

Sample standard deviation = s

Sample proportion – P (Capital)

Note: Parameters are related to population & statistics are related to sample.

2.13 SAMPLING DISTRIBUTION

Consider a population of size N and draw all possible samples of size n at random. For each sample we can compute the mean. The means of the samples will not be identical. If we group these different means according to their frequencies, the frequency distribution is formed is known as sampling distribution of the mean similarly we can have sampling distribution of the standard deviation etc.

While drawing each sample, we put back the previous sample so that the parent population remains the same. That is called sampling with replacement.

2.14 NUMBER OF POSSIBLE SAMPLES

Draw all possible samples of size n, from a given finite population of size N. Then the total number of all possible samples each of the same size n, which can be drawn from the population, is given by

$$N_{C_n} = \frac{\angle N}{\angle n \ \angle N-n}$$

In the sampling with replacement (infinite population)

The total number of samples with replacement is N^n.

2.15 STANDARD ERROR

The standard deviation of the sampling distribution is called standard error (S.E) thus the standard error of the sampling distribution of means is called standard error of means.

The reciprocal of the standard error is called precision

2.16 FINITE POPULATION: (σ known) (without replacement)

$$\mu_{\overline{X}} = \mu$$

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{N}} \sqrt{\frac{N-n}{N-1}} \quad \text{Where}$$

N $\to$ Size of the finite population

$\mu \to$ Mean of the population

$\sigma \to$ Standard deviation of the population

$\mu_{\overline{X}} \to$ Mean of the sampling distribution of mean

$\sigma_{\overline{X}} \to$ Standard deviation of the sampling distribution of mean and N>n

$\frac{N-n}{N-1}$ is known as the finite population correction factor.

2.17 INFINITE POPULATION: (σ KNOWN)

$$\mu_{\overline{X}} = \mu$$

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

Sampling is done with replacement.

Samples are drawn from an infinite population.

Standardized Sample Mean,

$$Z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$$

2.18 SAMPLING D

Suppose p be th

non occurrence of the

Now from an in

compute the proporti

Where $\mu_p \to$ Mean of

$\sigma_p^2 \to$ Varianc

If population is binon

The sampling distribut

Where N is the popul

2.19 SAMPLING DI

Suppose $\mu_{s_1} \to$

$\sigma_{s_1} \to$ Standard

For all possible s

Similarly $\mu_{s_2} \to$

$\sigma_{s_2} \to$ Standar

For all possible sampl

Now compute the stati

$\mu_{s_1 - s_2}$ →Mean of the

$\sigma_{s_1 - s_2}$ →Standard devi

Sampling distribution

Mean $\mu_{s_1 + s_2} = \mu_{s_1} +$

2.18 SAMPLING DISTRIBUTION OF PROPORTIONS

Suppose p be the probability of occurrence of an event then the probability of non occurrence of the event is $q = 1 - p$

Now from an infinite population, draw all possible samples of size n and then compute the proportion P of successes for each of the samples. Then

$$\mu_p = P \text{ and } \sigma_p^2 = \frac{pq}{n} = \frac{p(1-p)}{n}$$

Where $\mu_p \to$ Mean of the sampling distribution of proportions

$\sigma_p^2 \to$ Variance of the sampling distribution of proportions.

If population is binomially distributed,

The sampling distribution of proportion is normally distributed and if n is large. Then

$$\mu_p = P \text{ and } \sigma_p^2 = \frac{pq}{n} = \left(\frac{N-n}{N-1}\right)$$

Where N is the population size

2.19 SAMPLING DISTRIBUTION OF DIFFERENCES AND SUMS

Suppose $\mu_{s_1} \to$ mean of a sampling distribution of statistic S_1

$\sigma_{s_1} \to$ Standard deviation of sampling distribution of statistic S_1

For all possible samples of size n, drawn from population A

Similarly $\mu_{s_2} \to$ mean of a sampling distribution of statistic S_2

$\sigma_{s_2} \to$ Standard deviation of a sampling distribution of statistic S_2

For all possible samples of size n_2 drawn from population B.

Now compute the statistic $S_1 - S_2$

$\mu_{s_1-s_2} \to$ Mean of the sampling distribution of differences

$\sigma_{s_1-s_2} \to$ Standard deviation of the sampling distribution of differences

$$\mu_{s_1-s_2} = \mu_{s_1} - \mu_{s_2}$$

$$\sigma_{s_1-s_2} = \sqrt{\sigma_{s_1}^2 + \sigma_{s_2}^2}$$

Sampling distribution of sum of statistics

Mean $\mu_{s_1+s_2} = \mu_{s_1} + \mu_{s_2}$

Standard deviation $\sigma_{s_1+s_2} = \sqrt{\sigma^2_{s_1} + \sigma^2_{s_2}}$

For example

$$\mu_{\bar{X}_1+\bar{X}_2} = \mu_{\bar{X}_1} + \mu_{\bar{X}_2} = \mu_1 + \mu_2$$

$$\sigma_{\bar{X}_1+\bar{X}_2} = \sqrt{\sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2}} + \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Sampling distribution of differences of proportions

$$\mu_{p_1-p_2} = \mu_{p_1} - \mu_{p_2} = p_1 - p_2$$

$$\sigma_{p_1-p_2} = \sqrt{\sigma^2_{p_1} + \sigma^2_{p_2}} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$$

SOLVED EXAMPLES

Example 2.1. Out of 800 families with 5 children's each, how many would you expect to have

(Reg. Nov 2006 Set 2)
(April/May 2005 Reg. Set 3)
(Supple. Nov 2008.Set.2)

1. 3 boys
2. At least one boy

Assume equal probabilities for boys and girls.

Solution

Probability of boy = p (B) = p = $\frac{1}{2}$

And probability of girl = p (G) = q = $\frac{1}{2}$

Number of trials = n = 5

Number of boys in a family = X

1. Probability of a family having 3 boys

$$= (X = 3)$$

$$= 5_{C_3}\left(\frac{1}{2}\right)^3 \left(1 - \frac{1}{2}\right)^{5-3}$$

$$= \frac{\angle 5}{\angle 3 \ \angle 5-3} \cdot \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$= \frac{5 \times 4 \ \angle 3}{\angle 3 \ 2 \times 1} \left(\frac{1}{2}\right)^5$$

$$= 10 \times \frac{1}{32} = \frac{5}{16}$$

Expected number of families having 3 boys out of 5 children

$$= 800 \times \left(\frac{5}{16}\right) = 250$$

Means 250 families have 3 boys out of 5 children **Ans.**

2. At least one boy

At least one boy means at most 5 boys with 5 children's each

$$\sum_{x=0}^{5} b(x,\ n,\ p) = \sum_{x=0}^{5} b\left(x,\ 5,\ \frac{1}{2}\right) = 1.0000$$

$$\sum_{x=0}^{5} b(x,\ n,\ p) - p(x = 0)$$

$$= 1 - b\left(0,\ 5 \frac{1}{2}\right)$$

$$= 1 - 5_{C_0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0} = 1 - \left(\frac{1}{2}\right)^5$$

$$= \frac{31}{32} = 0.96875$$

Expected number of families having at least one boy

$$= 800 \times 0.96875$$

$$= 775$$

Ans.

Example 2.2. Find the maximum n such that the probability of getting no head in tossing a coin n times is greater than 0.1.

(Supple. Feb. 2007 Set 1)
(Reg. Nov. 2006 Set 3)
(Supple.Nov. /Dec. 2005 Set 3)

Solution

P = probability of getting head is = $\frac{1}{2}$

Given that p(x = 0 no head) > 0.1 = $\frac{1}{10}$

$$P(x=0) = n_{c_0}\, p^0 (1-p)^{n-0} > \frac{1}{10}$$

$$= \frac{\angle n}{\angle 0\ \angle n-o}\, 1\, (1-p)^n > \frac{1}{10}$$

$$\left(1-\frac{1}{2}\right)^n > \frac{1}{10}$$

$$\left(\frac{1}{2}\right)^n > \frac{1}{10}$$

For n = 1 probability (x = 0) = 0.5

n = 2 probability (x = 0) = 0.25

n = 3 probability = 0.125

n ≥ 4 probability is < 0.125

∴ n = 3 **Ans.**

Example 2.3. A burglar alarm system has six fail – safe component. The probability of each failing is 0.05 find these probabilities

(Reg. April/May 2004 Set 4)

a. Exactly three will fail

b. Fewer than two will fail

c. None will fail

Solution Given that p = 0.05, n = 6

b(x ; n, p) = b (x ; 6, 0.05)

1. Exactly three will fail x = 3

$$b(x;\, n,\, p) = b(3\,;6,\, 0.05) = n_{c_x} p^x q^{n-x}$$

$$= 6_{c_3}\, (0.05)^3 (1-0.05)^{6-3}$$

$$= 20\,(0.000125)\,(0.857375)$$

$$= 0.0021434$$ **Ans.**

2. Fewer than tw

$\sum_{x=0}^{1} b(x\,;\,n,\,p$

3. None will fail

$b(x;n,p) = b$

= 6

= 1

= 0

Example 2.4. A st
He guesses each an
of n that the probabi

Solution

$p(x \geq$

i.e.

$8_{C_0}\, p^0 (1-p)^{8-0} +$

$\left(\frac{1}{2}\right)^8$

2. Fewer than two will fail

$$\sum_{x=0}^{1} b(x\,;\,n,p) = b(0;6,0.05) + b(1;6,0.05)$$

$$= 6_{c_0}(0.05)\,(1-0.05)^{6-0} + 6_{c_1}(0.05)^1(1-0.05)^{6-1}$$

$$= 1 \times 1 \times (0.7350) + 6(0.05)(0.77378)$$

$$= 0.7350 + 0.232134$$

$$= 0.96713$$

Ans.

3. None will fail x = 0

$$b(x;n,p) = b(0;6,0.05)$$

$$= 6_{c_0}(0.05)^0(1-0.05)^{6-0}$$

$$= 1 \times 1 \times 0.7350$$

$$= 0.7350$$

Ans.

Example 2.4. A student takes a true false examination consisting of 8 questions. He guesses each answer. The guesses are made at random. Find the smallest value of n that the probability of guessing at least n correct answers is less than ½.

(Reg. April /May 2005 Set 4)

(Supple. Nov 2008.Set.3)

Solution

$$\text{Probability of guessing a question} = p = \frac{1}{2}$$

$$n = 8$$

$$p(x \geq n) = 1 - p(x=0) - p(x=0)\ldots\ldots p(x=n-1) < \frac{1}{2}$$

$$i.e.\ p(x=0) + p(x=1) + \ldots\ldots + p(x=n-1) > \frac{1}{2}$$

$$8_{C_0}\, p^0(1-p)^{8-0} + 8_{c_1}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{8-1} + \ldots\ldots + 8_{c_{n-1}}\left(\frac{1}{2}\right)^{n-1}\left(1-\frac{1}{2}\right)^{8-n+1} > \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^8 + 8_{c_1}\left(\frac{1}{2}\right)^8 + 8_{c_2}\left(\frac{1}{2}\right)^8 + \ldots\ldots + 8_{c_{n-1}}\left(\frac{1}{2}\right)^8 > \frac{1}{2}$$

$$8_{c_3}\left(\frac{1}{2}\right)^8 = \frac{\angle 8}{\angle 3\ \angle 5}\left(\frac{1}{2}\right)^8 = \frac{8\times 7\times 6}{3\times 2}\left(\frac{1}{2}\right)^8 = \frac{56}{256} = 0.218$$

$$8_{c_4}\left(\frac{1}{2}\right)^8 = \frac{\angle 8}{\angle 4\ \angle 4}\left(\frac{1}{2}\right)^8 = \frac{8\times 7\times 6\times 5}{4\times 3\times 2}\left(\frac{1}{2}\right)^8 = \frac{70}{256} = 0.2734$$

1. $\left(\frac{1}{2}\right)^8 = \frac{1}{256} = 0.00390$

2. $8_{c_1}\left(\frac{1}{2}\right)^8 = \frac{\angle 8}{\angle 1\ \angle 8-1}\left(\frac{1}{256}\right) = \frac{8}{256} = \frac{1}{128} = 0.00781$

3. $8_{c_2}\left(\frac{1}{2}\right)^8 = \frac{\angle 8}{\angle 2\ \angle 8-2}\left(\frac{1}{256}\right) = \frac{8\times 7}{2} = \frac{1}{256} = \frac{28}{256} = 0.109$

n-1 = 3, $\quad 0.00390 + 0.00781 + 0.109 + 0.218$

$= 0.3387$

n-1 = 4, $\quad 0.00390 + 0.00781 + 0.109 + 0.218 + 0.2734$

$= 0.6121 > \frac{1}{2}$

$n - 1 = 4$

$n = 5$

The least n, s.t. $p(x \geq n) < \frac{1}{2}$ is 5. **Ans.**

Example 2.5. Assume that 50% of all engineering students are good in mathematics. Determine the probabilities that among 18 engineering students

(Reg. April/May 2005)

(April/May 2004 Set 3)

1. exactly 10
2. at least 10
3. at most 8
4. at least 2 and at most 9 are good in mathematics

Solution

Let x = Number of engineering students who are good in math

1. Exactly 10

$p(x=10) =$

$= 18_{c_{10}}$

$= \frac{\angle}{\angle 10}$

$= \frac{18\times 17}{\angle}$

$= \frac{43758}{262144}$

$= 0.1669$

2. At least 10

At least 10 i.e

$= \sum_{x=10}^{18} 18_{c}$

$= 1 - \sum_{x=0}^{9}$

$= 1 - 0.5$

$= 0.4073$

3. At most 8

At most 8 mea

$p(x \leq 8) = \sum_{x=0}^{8}$

$= 0.4073$

4. At least 2 and

$$P = \text{Probability of good in math} = \frac{50}{100} = \frac{1}{2}$$

Total number of trials n = 18

1. Exactly 10

$$p(x=10) = b(x;n,p) = n_{c_x} p^x q^{n-x}$$

$$= 18_{c_{10}} \left(\frac{1}{2}\right)^{10} \left(1-\frac{1}{2}\right)^{18-10}$$

$$= \frac{\angle 18}{\angle 10\ \angle 18-10} \left(\frac{1}{2}\right)^{18}$$

$$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times \angle 10}{\angle 10\ \ 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2} \times \frac{1}{262144}$$

$$= \frac{43758}{262144}$$

$= 0.1669$ **Ans.**

2. At least 10

At least 10 i.e. at most 18

P (= At least 10) = P(x ≥ 10)

$$= \sum_{x=10}^{18} 18_{c_{10}} \left(\frac{1}{2}\right)^{10} \left(1-\frac{1}{2}\right)^{18-10}$$

$$= 1 - \sum_{x=0}^{9} 18_{c_x} \left(\frac{1}{2}\right)^{x} \left(1-\frac{1}{2}\right)^{18-x}$$

$$= 1 - 0.5927$$

$= 0.4073$ **Ans.**

3. At most 8

At most 8 means maximum 8

$$p(x \le 8) = \sum_{x=0}^{8} 18_{c_x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{18-x}$$

= 0.4073 from the binomial distribution table **Ans.**

4. At least 2 and at most 9, are good in math

$$p(2 \le x \le 9) = \sum_{x=2}^{9} 18_{c_x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{18-x}$$

$$= \sum_{x=0}^{9} b\left(x;18,\frac{1}{2}\right) - \sum_{x=0}^{1} b\left(x;18,\frac{1}{2}\right)$$

$$= 0.5927 - 0.0007$$

$$= 0.5920$$ **Ans.**

Example 2.6. 20% of items produced from a factory are defective. Find the probability that in sample of 5 chosen at random

(Reg. April/May 2005 Set 1)
(Supple. Nov 2008.Set.4)

1. None is defective
2. One is defective
3. $p(1 < x < 4)$

Solution

$$P \rightarrow \text{probability for the defective item } = \frac{20}{100} = \frac{1}{5}$$

$$n = 5$$

1. None is defective

$$p(x=0) = 5_{c_0}\left(\frac{1}{5}\right)^0 \left(1-\frac{1}{5}\right)^{5-0}$$

$$= 1 \times 1 \times \left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$$ **Ans.**

2. One is defective

$$p(x=1) = 5_{c_1}\left(\frac{1}{5}\right)^1 \left(1-\frac{1}{5}\right)^{5-1}$$

$$= \frac{\angle 5}{\angle 1 \; \angle 5-1} \times \frac{1}{5} \times \left(\frac{4}{5}\right)^4$$

$$= 5 \times \frac{1}{5} \times \frac{256}{625} = \frac{256}{625}$$ **Ans.**

3. $p(1 < x < 4) = p(x=2) + p(x=3)$

=

=

=

=

Example 2.7. Eac
pounds of newspa
two pounds. If a
generating

1. Between
2. More than
(Assume the

Solution
Given that

A Hyderabad house

Standard deviation i

The standard normal

$$= 5_{c_2}\left(\frac{1}{5}\right)^2\left(1-\frac{1}{5}\right)^{5-2} + 5_{c_3}\left(\frac{1}{5}\right)^3\left(1-\frac{1}{5}\right)^{5-3}$$

$$= 5_{c_2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 + 5_{c_3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2$$

$$= \frac{\angle 5}{\angle 2\,\angle 3}\ \frac{1}{25}\times\frac{64}{125} + \frac{\angle 5}{\angle 2\,\angle 3}\times\frac{1}{125}\times\frac{16}{25}$$

$$= \frac{20}{2}\times\frac{1}{25}\times\frac{64}{125} + \frac{20}{2}\times\frac{1}{125}\times\frac{16}{25}$$

$$= \frac{320}{6250}(4+1)$$

$$= \frac{1600}{6250}$$

$$= 0.256 \qquad \textbf{Ans.}$$

Example 2.7. Each month, a Hyderabad house hold generates on an average of 28 pounds of newspaper for garbage or recycling. Assume the standard deviation is two pounds. If a household is selected at random, find the probability of its generating

(Reg. April/May 2004 Set 4)

1. Between 27 and 31 pounds per month
2. More than 30.2 pounds per month

(Assume the variable is approximately normally distributed).

Solution

Given that

A Hyderabad house hold generates an average of 28 pounds.

$$\bar{x} = 28$$

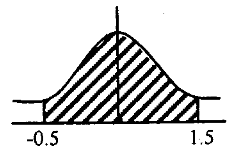
Standard deviation is two pounds

$$\sigma = 2$$

The standard normal variate $z = \dfrac{x-\bar{x}}{\sigma}$

1. Between 27 and 31 pounds per month

$$p(2 > \angle x < 31) = P\left(\frac{27-28}{2} < z < \frac{31-28}{2}\right)$$



$= p(-0.5 < z < 1.5)$

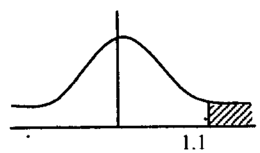
= Area from (0 to - 0.5) + Area from (0 to 1.5)

=0.1916+0.4332

=0.6248 **Ans.**

2. More than 30.2 pounds per month

$$p(x > 30.2) = P\left(z > \frac{30.2-28}{2}\right)$$



$= p(z > 1.1)$

= 0.5 – Area from (0 to 1.1)

= 0.5 – 0.3643

= 0.1357 **Ans.**

Example 2.8. If the probability of a bad reaction from a certain injection is 0.001. Determine the chance that out of 3,000 individuals more than two will get a bad reaction. **(Reg. April/May 2004 Set 3)**

Solution

n = 3000, p = .001

λ = np = 3000 × .001 = 3

Probability for more than two will get a bad reaction

$$p(x > 2) = 1 - p(x=0) - p(x=1) - p(x=2)$$

$$= 1 - \frac{e^{-\lambda}\lambda^x}{\angle x} - \frac{e^{-\lambda}\lambda^x}{\angle x} - \frac{e^{-\lambda}\lambda^x}{\angle x}$$

$$= 1 - \frac{e^{-3}3^0}{\angle 0} - \frac{e^{-3}3^1}{\angle 1} - \frac{e^{-3}3^2}{\angle 2}$$

$$= 1 - e^{-3} - 3e^{-3} - \frac{9}{2}e^{-3}$$

$$= 1 - .04978 - 3 \times 0.04978 - 4.5 \times 0.04978$$

= 1 - 0.04978 – 0.14934 – 0.22401

= 0.57687 **Ans.**

Example 2.9. If

1. What is
2. What i
probabi
3. What is

Solution

Suppose x = numl

Given that p = 0.3

n

1. The probab

$p(x=5) =$

$= 10_{c_5}$ (

= 252 (0.

= 0.1029

2. The most prol

= mean n

= n p = 1

Probability c

$p(x=3) =$

$= 10_{c_3}(0.3)$

$= 120 \times (0.0$

3. The probabilit

$p(x=10) =$

$= (0.3)$

= 0.000

Example 2.10. Fit
theoretical frequenc

Example 2.9. If the chance that on of the ten telephone lines is busy at an instant is 0.3

(Reg. April/May 2004 Set 2)

1. What is the chance that 5 of the lines are busy?
2. What is the most probable number of busy lines and what is the probability of this number?
3. What is the probability that all the lines are busy?

Solution

Suppose x = number of busy telephone lines

Given that p = 0.3,

n = 10 = number of telephone lines

1. The probability that 5 of the lines are busy

$$p(x=5) = 10_{c_5}(0.3)^5(1-0.3)^{10-5}$$

$$= 10_{c_5}(0.3)^5(0.7)^5$$

$$= 252\ (0.00243)\ (0-16807)$$

$$= 0.102919$$

Ans.

2. The most probable number of busy lines

= mean number of busy lines

= n p = 10 (.3) = 3

Probability of this

$$p(x=3) = 10_{c_3}(0.3)^3(1-0.3)^{10-3}$$

$$= 10_{c_3}(0.3)^3(0.7)^7$$

$$= 120 \times (0.027) \times (0.082359) = 0.26682$$

Ans.

3. The probability that all the lines are busy

$$p(x=10) = 10_{c_{10}}(0.3)^{10}(1-0.3)^{10-10}$$

$$= (0.3)^{10}$$

$$= 0.0000059049$$

Ans.

Example 2.10. Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones

(Reg. April/May 2004)

(Supple.2004)

x	0	1	2	3	4	5
f	2	14	20	34	22	8

Solution

n = 100, n = 5

$$\text{Mean } \bar{x} = \frac{\sum f_i x_i}{N}$$

$$= \frac{0\times2+1\times14+2\times20+3\times34+4\times22+5\times8}{2+14+20+34+22+8}$$

$$= \frac{14+40+102+88+40}{100}$$

$$= \frac{284}{100} = 2.84$$

Mean = n p = 2.84

5 p = 2.84

P = 0.568 $\qquad$ q = 1 – 9 = 1-0.568 = 0.432

$$x=0, p(x) = n_{c_0}\, p^0 q^5 = 5_{c_0}\,(0.568)^0(0.432)^5 = 0.01504$$

$$x=1, p(1) = n_{c_1}\, p^1 q^{n-1} = 5_{c_1}\,(0.568)^1(0.432)^{5-1} = 5\times(0.568)(0.432)^4 = 0.0989$$

$$x=2, p(2) = n_{c_2}\, p^2 q^{n-2} = 5_{c_2}\,(0.568)^2(0.432)^{5-2} = \frac{\angle 5}{\angle 2\ \angle 3}(0.568)^2(0.432)^3$$

$$= 10\times0.322624\times0.0806$$

$$= 0.26010$$

$$x=3, p(3) = n_{c_3}\, p^3 q^{n-3} = 5_{c_3}\,(0.568)^3(0.432)^{5-3} = \frac{\angle 5}{\angle 3\ \angle 5-3}(0.568)^3(0.432)^2$$

$$= 10\times0.1832\times0.18662$$

$$= 0.341895$$

$$x=4, p(4) = n_{c_4}\, p^4 q^{n-4} = 5_{c_4}\,(0.568)^4(0.432)^{5-4} = \frac{\angle 5}{\angle 4\ \angle 5-4}(0.568)^4(0.432)^3$$

$$= 5\times(0.10408)(0.432)$$

$$= 0.224826$$

$$x=5, p(5) = n_{c_5}\, p^5 q^{n-5} = 5_{c_5}\,(0.568)^5(0.432)^{5-5} = (0.568)^5 = 0.05912$$

Expected frequenc

N p(x) = 100 × p(0

Similarly

N p(x) = 100 × p(2

N p(x) = 100 × p(3

N p(x) = 100 × p(4

N p(x) = 100 × p(5

Example 2.11. Th

Determine the prob

1. none
2. one and
3. at least one w

Solution

Given that

The probability tha

p = 0.4

n = 5

1. None will gra

$b(x;n,p) = n_{c_x}$

$= 5_{C_(}$

$= (0.$

$= 0.($

2. One will gradu

$b(x;n,p) =$

$= 5_{C_1}$

$= \frac{}{\angle 1}$

$= 5\times$

$= 0.2$

Expected frequency

N p(x) = 100 × p(0) = 100 ×0.01504=1, Np(x) = 100×p(1) = 100 × 0.0989 = 9.89

Similarly

N p(x) = 100 × p(2) = 100 × 0. 26010 = 26.010

N p(x) = 100 × p(3) = 100 × 0.341895 = 34.189

N p(x) = 100 × p(4) = 100 × 0.224826 = 22.48

N p(x) = 100 × p(5) = 100 × 0.05912 = 5.912

Example 2.11. The probability that an entering student will graduate is 0.4. Determine the probability that out of 5 students **(Reg. April/May 2004 Set 1)**

1. none
2. one and
3. at least one will graduate

Solution

Given that

The probability that on entering student will graduate is 0.4.

p = 0.4

n = 5

1. None will graduate i.e. x = 0

$$b(x;n,p) = n_{C_x}\ p^x q^{n-x}$$

$$= 5_{C_0}\ (0.4)^0 (1-0.4)^{5-0}$$

$$= (0.6)^5$$

$$= 0.07776$$

Ans.

2. One will graduate i.e. x = 1

$$b(x;n,p) = n_{C_x}\ p^x q^{n-x}$$

$$= 5_{C_1}\ (0.4)^1 (1-0.4)^{5-1}$$

$$= \frac{\angle 5}{\angle 1\ \angle 5-1}(0.4)(0.6)^4$$

$$= 5 \times (0.9)(0.1296)$$

$$= 0.2592$$

Ans.

3. At least one will graduate i.e. at most 5 can graduate

$$p(1 \le x \le 5) = \sum_{x=1}^{5} 5_{C_x} (0.4)^x (1-0.4)^{5-x}$$

$$= \sum_{x=0}^{5} 5_{C_x} (0.4)^x (0.6)^{5-x} - p(x=0)$$

$$= 1.000 - 0.07776$$

$$= 0.92224$$ **Ans.**

Example 2.12 Fit a Poisson distribution to calculate the theoretical frequencies for the following data **(Supple. Nov/Dec. 2004 Set 2)**

x	0	1	2	3	4
f	109	65	22	3	1

Solution

N = 200

Mean of Poisson distribution = λ

$$\lambda = \frac{\sum f_i x_i}{\sum f_i} = \frac{0\times 109 + 1\times 65 + 2\times 22 + 3\times 3 + 4\times 1}{109+65+22+3+1}$$

$$= \frac{0+65+44+9+4}{200} = \frac{122}{200} = 0.61$$

$$p(0) = \frac{e^{-\lambda}\lambda^x}{\angle x} = \frac{e^{-0.61}(0.61)^0}{\angle 0} = \frac{0.5433\times 1}{1} = 0.5433$$

$$f(x) = N \times p(x)$$

f(0) = 200 × p(0) = 200 × 0.5433 = 108.66

f(1) = 200 × p(1)

f (x+1) = N p (x +1)

$$= \frac{N.\lambda}{x+1} p(x)$$

$$= \frac{\lambda}{x+1} N.P(x)$$

$$f(x+1) = \frac{\lambda}{x+1} f(x)$$

$$f(1) = \frac{0.61}{0+1} f(0) = \frac{0.61}{1}(108.66) = 66.2826$$

$$f(2) = \frac{0.61}{2} f(1) =$$

$$f(3) = \frac{0.61}{3} f(2) =$$

$$f(4) = \frac{0.61}{4} f(3) =$$

Example 2.13. Fou

occur is given below

Nc

Fit a binomial distrib

Solution

Probability of getting

$$p(x) =$$

$$f(2)=\frac{0.61}{2}f(1)=\frac{0.61}{2}\times 66.2826=20.198$$

$$f(3)=\frac{0.61}{3}f(2)=\frac{0.61}{3}\times 20.198=4.1069$$

$$f(4)=\frac{0.61}{4}f(3)=\frac{0.61}{4}\times 4.1069=0.6263$$

Example 2.13. Four coins are tossed 160 times. The number of times X heads occur is given below **(Supple. Nov/Dec. 2004)**

X	0	1	2	3	4
No. of times	8	34	69	43	6

Fit a binomial distribution to this data on the hypothesis that coins are unbiased

Solution

Probability of getting a head = 1/2

$$p=\frac{1}{2},\ q=\frac{1}{2}$$

$$p(x)=n_{C_x}\ p^x q^{n-x} \quad , \quad p(x+1)=n_{C_{x+1}}p^{x+1}q^{n-x-1}$$

$$n = 4,\ N = 140$$

$$p(x+1)=\frac{(n-x)}{(x+1)}\left(\frac{p}{q}\right).p(x)$$

$$p(0)=4_{C_x}(p)^x(q)^{n-x}$$

$$=4_{C_0}(p)^0(q)^{n-0}$$

$$=4_{C_0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{4-0}=\left(\frac{1}{2}\right)^4$$

$$\text{P }(0)=\left(\frac{1}{2}\right)^4$$

$$p(1)=4_{C_1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{4-1}$$

$$= 4.\frac{1}{2}\left(\frac{1}{2}\right)^3$$

$$p(1) = 4\left(\frac{1}{2}\right)^4$$

$$f(0) = N.p(0) = 160\times\left(\frac{1}{2}\right)^4 = 160\times\frac{1}{16} = 10$$

$$f(1) = 4\left(\frac{1}{2}\right)^4.160 = 4\times\frac{1}{16}\times 160 = 40$$

$$f(2) = \frac{4-1}{1+1}f(1) = \frac{3}{2}\times 40 = 60,\ n = 4,\ x = 1$$

$$f(3) = \frac{4-2}{1+2}f(2) = \frac{2}{3}\times 60 = 40, n = 4, x = 2$$

$$f(4) = \frac{4-3}{1+3}f(3) = \frac{1}{4}.40 = 10$$

Expected frequencies or 10, 40, 60, 40, 10

Example 2.14 If X is a Poisson variate such that
P(x = 0) = p(x = 2) + 3 p (x = 4), find

(Supple. Nov. / Dec. 2004)

1. The mean of x
2. $P(x \le 2)$

Solution

Given that

$$P(x = 0) = p(x = 2) + 3\,p\,(x = 4)$$

$$\frac{e^{-\lambda}\lambda^0}{\angle 0} = \frac{e^{-\lambda}\lambda^2}{\angle 2} + 3\times\frac{e^{-\lambda}\lambda^4}{\angle 4}$$

$$p(x,\lambda) = \frac{e^{-\lambda}\lambda^x}{\angle x}$$

$$1 = \frac{\lambda^2}{2} + \frac{3\lambda^4}{24}$$

1 = 12

$12\lambda^2 + 3\lambda$

$\lambda^4 + 4\lambda^2 = 8$

$\lambda^4 + 4\lambda^2 - 8 =$

$\lambda^2 = \frac{-4 \pm \sqrt{1}}{}$

$\lambda^2 = \frac{-4 \pm \sqrt{1}}{2}$

$\lambda^2 = \frac{-4 \pm 6.}{2}$

$\lambda^2 = \frac{-4 + 6.}{2}$

$\lambda^2 = -5.4641$

2) $p(x \le 2) =$

=

=

=

=

Example 2.15. If X is

P (x = 0) = p (

Using recurrenc

Solution

Given that

Probability (x = 0) = P

$$1 = \frac{12\lambda^2 + 3\lambda^4}{2\lambda}$$

$$12\lambda^2 + 3\lambda^4 = 24$$

$$\lambda^4 + 4\lambda^2 = 8$$

$$\lambda^4 + 4\lambda^2 - 8 = 0$$

$$\lambda^2 = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times (-8)}}{2}$$

$$\lambda^2 = \frac{-4 \pm \sqrt{16+32}}{2} = \frac{-4 \pm \sqrt{48}}{2}$$

$$\lambda^2 = \frac{-4 \pm 6.9282}{2}$$

$$\lambda^2 = \frac{-4 + 6.9282}{2} \; OR \; \frac{-4 - 6.9282}{2}$$

$$\lambda^2 = \;-5.4641 \quad \text{OR} \quad +1.4641$$

2) $$p(x \le 2) = \sum_{x=0}^{2} \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= e^{-\lambda} + \frac{e^{-\lambda}\lambda}{\lfloor 1} + \frac{e^{-\lambda}\lambda^2}{\lfloor 2}$$

$$= e^{-1.4641} + \frac{e^{-1.4641}(1.4641)}{\lfloor 1} + \frac{e^{-1.4641}(1.4641)^2}{\lfloor 2}$$

$$= 0.23132 + 0.3386 + 0.24792$$

$$= 0.81784$$

Example 2.15. If X is a Poisson variate such that

P (x = 0) = p (x = 1), find p (x = 0) and

Using recurrence formulae find the Probabilities at x = 1, 2, 3, 4 and 5

(Supple. Nov/Dec 2004)

Solution

Given that

Probability (x = 0) = Probability (x = 1)

$$p(x,\lambda)=\frac{e^{-\lambda}\lambda^{x}}{\angle x}$$

$$\frac{e^{-\lambda}\lambda^{0}}{\angle 0}=\frac{e^{-\lambda}\lambda^{1}}{\angle 1}$$

$$e^{-\lambda}=e^{-\lambda}.\lambda$$

$$\lambda = 1$$

$$\text{Probability at } (x=0)=\frac{e^{-\lambda}\lambda^{x}}{\angle x}=\frac{e^{-1}(1)^{0}}{\angle 0}=e^{-1}=0.367$$

Probability at (x = 1) = 0.3679

Recurrence relation

$$p(x+1)=\frac{\lambda}{x+1}p(x)$$

Put x = 1 $p(2)=\frac{1}{2}p(1)=\frac{1}{2}0.3679=0.18395$ **Ans.**

$x=2,\ p(3)=\frac{1}{3}P(2)=\frac{1}{3}\times 0.18395=0.0613166$ **Ans.**

$x=3,\ p(4)=\frac{1}{4}p(3)=0.015329$ **Ans.**

$x=4,\ p(5)=\frac{1}{5}p(4)=0.003065$ **Ans.**

Example 2.16. In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5. Assuming the distribution to be normal, find

(Supple. Nov/Dec 2004 Set 4)

1. How many students score between 12 and 15?
2. How many score above 18
3. How many score below 8

Solution

Given that

Mean of a certain test is 14

Standard deviation is 2.5

Assuming the distribution to be normal $z=\frac{x-\bar{x}}{\sigma}$

1. How many st

$p(12<x<15$

= Area

= Area

= 0.031

= 0.187

Number of st

2. How many sc

$p(x>18)=$

$=p(z>1.6)$

$=0.5-area$

= 0.5 – 0.445

Number of st

3. How many sc

$p(x<8)=$

= 0.5 –

= 0

= 0.008

Example 2.17. If a

P (x = 1). $\frac{3}{2}$ =

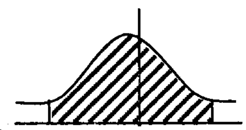
Find

1. $P(x\geq 1)$
2. $p(x\leq 3)$
3. $p(2\leq x\leq 5)$

1. How many students score between 12 and 15 =?

$$p(12 < x < 15) = P\left(\frac{12-14}{2.5}\right) < z < \frac{15-14}{2.5}$$

$$= p(-0.8 < z < 0.4)$$



= Area from (0 to - 0.8) + Area from (0 to 0.4)

= Area from (0 to 0.8) + Area from (0 to 0.4)

= 0.0319 + 0.1554

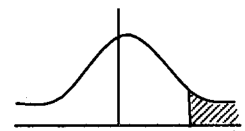
= 0.1873

Number of students = 1000 × 0.1873 = 187.3

≈ 187 **Ans.**

2. How many score above 18

$$p(x > 18) = p\left(z > \frac{18-14}{2.5}\right)$$



$$= p(z > 1.6)$$

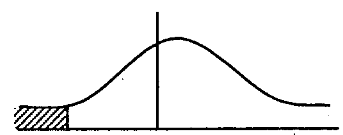
$$= 0.5 - area \;\; from (0 \;\; to \;\; 1.6)$$

= 0.5 – 0.4452 = 0.0548

Number of students = 1000 × 0.0548 = 54.8 ≈ 55 **Ans.**

3. How many score below 8

$$p(x < 8) = p\left(z < \frac{8-14}{2.5}\right) = p(z < -2.4)$$



= 0.5 – area from (0 to 2.4)

= 0.5 – 0.4918

= 0.0082 **Ans.**

Example 2.17. If a Poisson distribution is such that

$$P(x = 1).\frac{3}{2} = p(x = 3)$$ **(Reg. April/May 2005)**

Find **(Reg. Nov 2006 Set 1)**

1. $P(x \geq 1)$
2. $p(x \leq 3)$
3. $p(2 \leq x \leq 5)$

Solution

Given that $$p(x, \lambda) = \frac{e^{-\lambda}\lambda^x}{\angle x}$$

$$p(x=1)\frac{3}{2} = p(x=3)$$

$$\frac{e^{-\lambda}\lambda^1}{\angle 1} \times \frac{3}{2} = \frac{e^{-\lambda}\lambda^3}{\angle 3}$$

$$\frac{3\lambda}{2} = \frac{\lambda^3}{6}$$

$$9\lambda = \lambda^3$$

$$\lambda(9-\lambda^2) = 0$$

$$\lambda = 0 \ or \ 9 - \lambda^2 = 0$$

$$\lambda = \pm 3$$

1. $p(x \geq 1) = 1 - p(x=0)$

$$= 1 - \frac{e^{-\lambda}\lambda^x}{\angle x} = 1 - \frac{e^{-3}3^0}{\angle 0} = 1 - e^{-3} = 1 - 0.049$$

$= 0.95019$ **Ans.**

2. $p(x \leq 3) = \sum_{x=0}^{3} \frac{e^{-\lambda}\lambda^x}{\angle x}$

$$= \frac{e^{-3}\lambda^x}{\angle x} = 0.647$$ **Ans.**

3. $p(2 \leq x \leq 5) = \sum_{x=0}^{5} \frac{e^{-\lambda}\lambda^x}{\angle x} - \sum_{x=0}^{2} \frac{e^{-\lambda}\lambda^x}{\angle x} \ [\lambda = 3]$

$$= 0.916 - 0.423$$

$= 0.493$ **Ans.**

Example 2.18. A sales tax officer has reported that the average sales of the 500 business that he has to deal with during a year are Rs. 36,000 with a standard deviation of 10,000. Assuming that the sales in these businesses are normally distributed. Find

(Reg. April/May 2005 Set 1)

(Reg. Nov 2006 Set 1)

1. The number of
2. The percentage
Rs. 30,000/- ar

Solution

Given that

Total number of busi

Mean = 36,000

Standard deviation =

Standard random vari

1. The number of

p(

50

2. The percentage

[Rs. 30,000/- and Rs.4

$p(30{,}000 < x <$

$= p(-0.6 < z$

= Area from (0

$= 0.2258 + 0.15$

$= 0.3812$

500 x 0.3812=19

Example 2.19. Using

4 and 5. If the mean of

Solution

1. The number of business as the sales of while are Rs. 40,000/-
2. The percentage of business the sales of while are likely to range between Rs. 30,000/- and Rs. 40,000/-

Solution

Given that

Total number of business = 500

Mean = 36,000

Standard deviation = 10,000

Standard random variable $z = \dfrac{x - \bar{x}}{\sigma}$

1. The number of business as the sales of while are Rs. 40,000

$$p(x > 40{,}000) = p\left(z > \frac{40{,}000 - 36{,}000}{10{,}000}\right)$$

$$= p\,(z > 0.4)$$

$$= 0.5 - 0.1554$$

$$= 0.3446$$

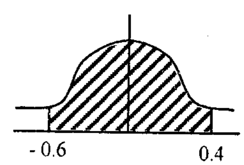
500 X 0.3446 = 172.3 Business

2. The percentage of business the sales of while are likely to range between [Rs. 30,000/- and Rs.40, 000/-)

$$p(30{,}000 < x < 40{,}000) = p\left(\frac{30{,}000 - 36{,}000}{10{,}000} < z < \frac{40{,}000 - 36{,}000}{10{,}000}\right)$$

$$= p(-0.6 < z < 0.4)$$

= Area from (0 to – 0.6) + area from (0 to 0.4)



$= 0.2258 + 0.1554$

$= 0.3812$

500 x 0.3812=190.6

Ans.

Example 2.19. Using Recurrence formula find the probabilities when x = 0, 1, 2, 3, 4 and 5. If the mean of Poisson distribution is 3.

Solution

Given that $\lambda = 3$

$$p(x, \lambda) = \frac{e^{-\lambda}\lambda^{x}}{\angle x}$$

$$\text{Recurrence Relation} \quad p(x+1) = \frac{\lambda}{x+1}\; p(x)$$

$$x = 0,\; p(0) = p(0,3) = \frac{e^{-3}3^{0}}{\angle 0} = e^{-3} = 0.04978$$

$$x = 0,\; p(1) = \frac{\lambda}{x+1} p(0) = \frac{3}{1} p(0) = \frac{3}{1} \times (0.04978) = 0.1494$$

$$x = 1,\; p(2) = \frac{\lambda}{x+1} p(1) = \frac{3}{2} p(1) = \frac{3}{2} \times (0.1494) = 0.22404$$

$$x = 2 \;\; p(3) = \frac{\lambda}{x+1} p(2) = \frac{3}{3} p(2) = \frac{3}{3} \times (0.22404) = 0.22404$$

$$x = 3 \;\; p(4) = \frac{\lambda}{x+1} p(3) = \frac{3}{4} p(3) = \frac{3}{5} \times (0.22404) = 0.13442$$

$$x = 4 \;\; p(5) = \frac{\lambda}{x+1} p(4) = \frac{3}{5} p(4) = \frac{3}{6} \times (0.13442) = 0.067212$$

Example 2.20. If the masses of 300 students are normally distributed with mean 68 k.gms. and standard deviation 3 k.gms. How many students have masses?

(Reg. April/May 2005 Set 4)

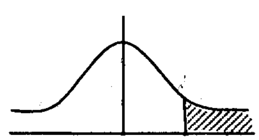
1. Greater than 72 k.gms.
2. Less than or equal to 64k.gms.
3. Between 65 and 70 k.gms. inclusive

Solution

Mean $\bar{x} = 68$ k.gms.

Standard deviation $\sigma = 3$ k.gms.

Greater than 72 k.gms.



$$z = \frac{X - \bar{X}}{\sigma} = \frac{72 - 68}{\sigma} = 1.333$$

$$p(x \geq 72) = p(x \geq 1.333)$$

$$= 0.5 - 0.4082$$

$$= 0.0918$$

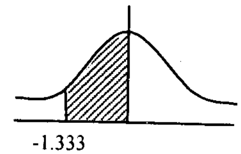
Number of students = 300×0.0918

$= 27.54 = 28$ approx. **Ans.**

1. Less than or equal to 64 k.gms.

$$z = \frac{X - \overline{X}}{\sigma} = \frac{64 - 68}{3} = -1.333$$

$$p(x \le 64) = p(Z \le -1.333)$$



$$= 0.4082 \quad \textbf{Ans.}$$

Number of students

$$= 300 \times 0.4082$$

$$= 122.40 = 123 \text{ Approx.}$$

2. Between 65 and 70 k.gms. inclusive

$$z = \frac{X - \overline{X}}{\sigma}$$

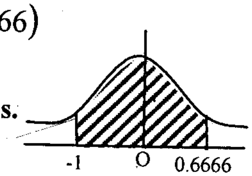
$$= \frac{65 - 68}{3}, \frac{70 - 68}{3}$$

$$z = -1,\ 0.6666$$

$$p(65 \le x \le 70) = p(-1 \le z \le 0.6666)$$

$$= 0.3413 + 0.2454$$

$$= 0.5867 \quad \textbf{Ans.}$$



Number of students

$$= 300 \times 0.5867$$

$$=176.01=177 \text{approx.} \quad \textbf{Ans.}$$

Example 2.21. A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the probability that a particular packet will violate the guarantee. **(Nov / Dec 2005 Set 4)**

Solution

The probability of a seed not germinating

$$p = \frac{5}{100} = 0.05$$

Let x = R.V = number of seeds that do not germinate

λ = mean number of seeds, in a sample of 200, which do not germinate

= n p

= 200 × 0.05 = 10

A packet will violate guarantee if it contains more than 20 non – germinating seeds.

Probability that the guarantee is violated

$$p(X > 20)$$
$$= 1 - p(X \le 20)$$
$$= 1 - \sum_{x=0}^{20} \frac{3^{-10}10^{x}}{\angle x}$$
$$= 1 - F(20,10)$$
$$= 1 - 0.998$$
$$= 0.002$$

Ans.

Example 2.22. If the mean and S.D. of normal distribution are 70 and 16, find p (38 < x < 46)

(Nov. / Dec. 2005 Set 4)

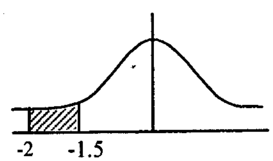
Solution

Given that $\overline{X} = 70$

Standard deviation $\sigma = 16$

$p(38) < x < 46 = ?$

$$z = \frac{X - \overline{X}}{\sigma}$$
$$= \frac{38-70}{16}, \frac{46-70}{16}$$
$$= \frac{-32}{16,} \frac{-24}{16}$$
$$z = -2, -1.5$$



$$p(38 < x < 46) = p(-2 < z < -1.5)$$

= area from (0 to -2) – area from (0 to -1.5)

= area from (0 to 2) – Area from (0 to 1.5)

= 0.4772 – 0.4332

= 0.044 **Ans.**

Example 2.23. If X is a normal variate, find the probability

(Supple. Feb. 2007 Set 2, Set 4)

1. to the left of z = - 1.78

2. to the righ
3. Correspon
4. to the left

Solution

1. To the left
A = (
= (
= (

2. To the right
A = 0.5 + Area fr

3. Correspondi
A = Area from (0
= Area fro
= 0.4370 + 0.28
= 0.7251

4. To the left of
A = {0.5 –
= (0.5 – 0.4
= 0.0059 + (
= 0.0395

Example 2.24. T
respectively find

Solution

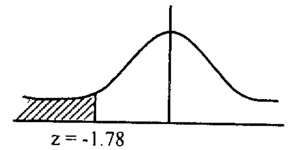
Given that

Mean $\overline{X} = 8$

2. to the right of $z = -1.45$
3. Corresponding to $-0.80 \le z \le 1.53$
4. to the left of $z = -2.52$ and to the right of $z = 1.83$

Solution

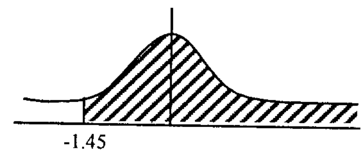
1. To the left of $z = -1.78$



$$A = 0.5 - (\text{Area } 0 \text{ to } -1.78)$$
$$= 0.5 - (\text{Area } 0 \text{ to } 1.78)$$
$$= 0.5 - 0.4625$$
$$= 0.375$$

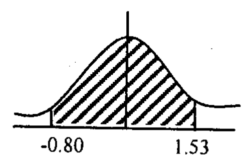
Ans.

2. To the right of $z = -1.45$



$$A = 0.5 + \text{Area from } 0 \text{ to } 1.45$$
$$= 0.5 + \text{area from } 0 \text{ to } 1.45$$
$$= 0.5 + 0.4265$$
$$= 0.9265$$

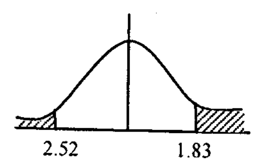
3. Corresponding to $-0.80 \le z \le 1.53$



$$A = \text{Area from } (0 \text{ to } -0.80)$$
$$= \text{Area from } (0 \text{ to } 0.80) + \text{Area from } (0 \text{ to } 1.53)$$
$$= 0.4370 + 0.2881$$
$$= 0.7251$$

Ans.

4. To the left of $z = -2.52$ and to the right of $z = 1.83$



$$A = \{0.5 - \text{Area from } (0 \text{ to } 2.52)\}$$
$$+ [0.5 - \text{Area (from } 0 \text{ to } 1.83)]$$
$$= (0.5 - 0.4941) + (0.5 - 0.4664)$$
$$= 0.0059 + 0.0336$$
$$= 0.0395$$

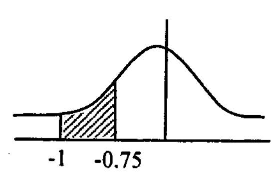
Ans.

Example 2.24. The mean and standard deviation of a normal variate are 8 and 4 respectively find **(Nov. / Dec. 2005 Set 2)**

1. $p(5 \le x \le 10)$ 2. $p(x \ge 5)$

Solution

Given that

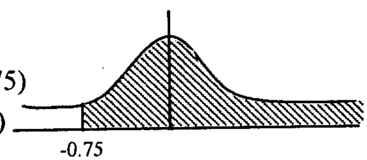


Mean $\overline{X} = 8$

Standard deviation $\sigma = 4$

1. $$p(5 \le x \le 10) = p\left(\frac{5-8}{4} < z < \frac{4-8}{4}\right)$$
$$= p(-0.75 < z < -1)$$
$$= \text{Area from } (0 \text{ to } 1) - \text{area from } (0 \text{ to } 0.75$$
$$= \text{Area from } (0 \text{ to } 1) - \text{area from } (0 \text{ to } 0.75)$$
$$= 0.3413 - 0.2734$$
$$= 0.0679$$ **Ans.**

2. $$p(X \ge 5) = p\left(z \ge \frac{5-8}{4}\right)$$
$$= p(z \ge .0.75)$$
$$= 0.5 + \text{Area from } (0 \text{ to } -0.75)$$
$$= 0.5 + \text{Area from } (0 \text{ to } 0.75)$$
$$= 0.5 + 0.2734$$
$$= 0.7734$$ **Ans.**



Example 2.25. Find the probability that at most 5 defective components will be found in a lot of 200 it experience. Show that 2% of such components are defective. Also find the probability of more than five defective components.

(Nov/Dec 2005 Set 4)
(Supple. Nov.2008)

Solution Given that,

Probability of defective components $p = \frac{2}{100} = .02$

n = 200

mean $\lambda = 200 \times 0.02 = 4$

1. Probability that at most 5 defective components

$$= p\ (x \le 5)$$
$$= p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4) + p(x=5)$$
$$p(x, \lambda) = \frac{e^{-\lambda}\lambda^x}{\angle x}$$
$$= \frac{e^{-4}(4)^0}{\angle 0} + \frac{e^{-4}(4)^1}{\angle 1} + \frac{e^{-4}(4)^2}{\angle 2} + \frac{e^{-4}(4)^3}{\angle 3} + \frac{e^{-4}(4)^4}{\angle 4} + \frac{e^{-5}(4)^5}{\angle 5}$$

$= e^{-4}\Big[1 + 4 \cdot$

$= 0.01832$ [1

$= 0.7853$

2. Probability of r

$= 1\text{-}P(X \le 5)$

$= 1\text{-}0.7853$

$= 0.2147$

Example 2.26. If

$3p(x=4) =$

1. The mean

Solution

Given that

3

$$= e^{-4}\left[1+4+8+\frac{32}{3}+\frac{32}{3}+\frac{128}{15}\right]$$

$= 0.01832$ [1 + 4 + 8 + 10.666 + 10.666 + 8.5333]

$= 0.7853$ **Ans.**

2. Probability of more than five defective components

$= 1 - P(X \leq 5)$

$= 1 - 0.7853$

$= 0.2147$ **Ans.**

Example 2.26. If X is a Poisson variate such that **(Feb. 2007 Set 3)**

$$3p(x=4) = \frac{1}{2}p(x=2) + p(x=0) \text{ Find}$$

1. The mean of x 2. $p(X \leq 2)$

Solution

Given that

$$3p(x=4) = \frac{1}{2}p(x=2) + p(x=0)$$

$$3.\frac{e^{-\lambda}\lambda^x}{\angle x} = \frac{1}{2}\frac{e^{-\lambda}\lambda^x}{\angle x} + \frac{e^{-\lambda}\lambda^x}{\angle x}$$

$$3.\frac{e^{-\lambda}\lambda^4}{\angle x} = \frac{1}{2}\frac{e^{-\lambda}\lambda^2}{\angle x} + \frac{e^{-\lambda}\lambda^0}{\angle x}$$

$$3\frac{\lambda^4}{24} = \frac{1}{2}\frac{\lambda^2}{2} + 1$$

$$\frac{\lambda^4}{8} = \frac{\lambda^2}{4} + 1$$

$$\frac{\lambda^4}{8} = \frac{\lambda^2+4}{4}$$

$$\lambda^4 = 2\lambda^2 + 8$$

$$\lambda^4 - 2\lambda^2 - 8 = 0$$

$$\lambda^2 = \frac{2 \pm \sqrt{4+4\times1\times(-8)}}{2}$$

$$= \frac{2 \pm \sqrt{4+32}}{2}$$

$$\lambda^2 = \frac{2 \pm 6}{2}$$

$$\lambda^2 = \frac{2+6}{2}, \frac{2-6}{2}$$

$$\lambda^2 = 4, -2$$

$$\lambda = \pm 2$$

$$\lambda = 2$$

1. The mean of x

   Mean $\lambda = 2$

2. $p(x \le 2) = \sum_{x=0}^{2} \frac{e^{-\lambda} \lambda^x}{\angle x} = \sum_{x=0}^{2} \frac{e^{-2} 2^x}{\angle x} = 0.677$ **Ans.**

Example 2.27. The marks obtained in mathematics by 1000 students is normally distributed with mean 78% and standard deviation 11% determine

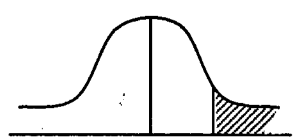
(Supple. Feb 2007 Set 1)

1. How many students got marks above 90%
2. What was the highest mark obtained by the lowest 10% of the student

Solution

Let X → marks of the students

The standard normal variate is $z = \frac{x-\mu}{\sigma}$



1.0909

Given that mean $\mu = 78\% = 0.78$

Standard deviation $\sigma = 11\% = 0.11$

1. Probability for the students who got marks above 90%

$$P(x > 0.90) = p\left(z > \frac{0.90 - 0.78}{0.11}\right)$$

$$= p\left(z > \frac{0.12}{0.11}\right)$$

$$= p(z > 1.0909)$$

$$= 0.5 - \text{Area from } (0 \text{ to } 1.0909)$$

No. of stud

2. What was t

$z = \frac{\overline{X} - \mu}{\sigma}$

For the low

Means z_1 m

Example 2.28.
board between 2
particular minute t

1. 4 or few

Solution

$\lambda = 2.5$

Let x = R.V

1. 4 or fewer

$p(x$

2. more than 6 c

$p(x > 6) = 1 -$

$= 1 -$

$= 0.5 - 0.3621$

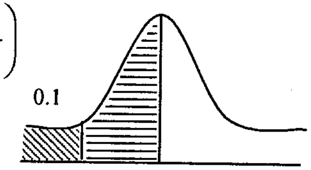
$= 0.1379$

No. of students $= 1000 \times 0.1379 = 137.9 \approx 138$

2. What was the highest mark obtained by the lowest 10% of the student

$z = \dfrac{\overline{X} - \mu}{\sigma}$ 10% Of the students i.e. 100 students.

For the lowest 10% students' area will be $0.1\left(< \dfrac{1}{2}\right)$



Means z_1 must be negative

0.5 – (area from 0 to 0.1)

$= 0.5 - 0.0398$

$= 0.401$

Example 2.28. The average number of phone calls / minute coming into a switch board between 2 pm and 4pm is 2.5. Determine the probability that during one particular minute there will be **(Supple. Feb. 2007 Set 1)**

1. 4 or fewer 2. more than 6 calls

Solution

$\lambda = 2.5$

$$f(x;\lambda) = f(x;2.5)$$

$$= \frac{(2.5)^x e^{-2.5}}{\angle x}$$

Let x = R.V = number of phone calls / minute during 2 pm and 4pm.

1. 4 or fewer

$$p(x \le 4) = \sum_{x=0}^{4} f(x;\ 2.5)$$

$$= F(4;\ 2.5) \qquad \left[\frac{0.904 + 0.877}{2} = 0.8905\right]$$

$$= 0.8905$$

Ans.

2. more than 6 calls

$$p(x > 6) = 1 - p(x \le 6)$$

$$= 1 - \sum_{x=0}^{6} f(x;2.5)$$

$$= 1 - F(6; 2.5) \qquad \left[\frac{0.988 + 0.983}{2} = 0.9855\right]$$

$$= 1 - 0.9855$$

$$= 0.0145$$

Ans.

Example 2.29 Find the mean and standard deviation of a normal distribution in which 7% of items are under 35 and 89% are under 63. **(Reg. Nov 2006 Set2, Set3, Set4)**

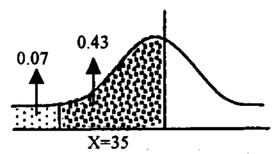
(Nov.2009 set 4)

Solution

Given that 7% of items are under 35 and 89% of items are under 63

Mean =?

Standard deviation =?



1) $p(x < 35) = 0.07 < \frac{1}{2}$

$\Rightarrow$ Z must be negative such that area from 0 to z is

0.5 – 0.07 = 0.43

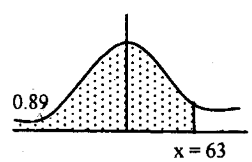
From normal distribution table z = - 1.48

2) $p(x < 63) = 0.89 < \frac{1}{2}$

Z must be positive such that area from 0 to z is 0.89 – 0.5 = 0.39

From normal distribution table z = 1.23

We have



$$z = \frac{x - \bar{x}}{\sigma}$$

$$-1.48 = \frac{35 - \bar{x}}{\sigma} \Rightarrow \bar{x} - 1.480 = 35 \text{........}(A)$$

$$1.23 = \frac{63 - \bar{x}}{\sigma} \Rightarrow \bar{x} + 1.23\sigma = 63$$

By solving (A) and (B)

Arithmetic mean $\bar{x} = 50.3$ **Ans.**

Standard deviation σ = 10.33 **Ans.**

Example 2.30. Th
day. The no. of acci
Calculate

Solution

f (x

P (

Example 2.31. A
The numbers of
distribution. Calcul

Solution

λ = 3, x = 1

P (× = x)

P(× =1)

Example 2.32. On
the rate of 5 per
distributed accordin
exactly 7 emissions

Solution

P(× = x)

P (× = 7

Example 2.30. The traffic police recorded on an average of 4 road accidents per day. The no. of accidents is distributed according to a Poisson distribution.

Calculate the probability in any day of exactly 2 accidents

Solution

$$f(x, \lambda) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{\lfloor x}$$

$$P(X = 2) = \frac{(4)^2\ e^{-4}}{\lfloor 2}$$

$$= \frac{16 \times 0.0183}{2} \qquad = 0.1465 \qquad \textbf{Ans.}$$

Example 2.31. A printing press recorded on an average of 3 mistakes per page. The numbers of printing mistakes are distributed according to a Poisson distribution. Calculate the probability in any page of exactly one mistake.

Solution

$\lambda = 3, \quad x = 1$

$$P(\times = x) = \frac{\lambda^x e^{-\lambda}}{\lfloor x}$$

$$P(\times = 1) = \frac{3^1\ e^{-3}}{\lfloor 1}$$

$$= 3e^{-3}$$

$$= 3 \times 0.0497$$

$$= 0.1493 \qquad \textbf{Ans.}$$

Example 2.32. On the average, alpha particles are emitted by ratio active source at the rate of 5 per every minute. It is given that the numbers of particles are distributed according to the Poisson distribution. Calculate the probability of getting exactly 7 emissions in one minute

Solution

$$P(\times = x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$P(\times = 7) = \frac{e^{-5}\,(5)^7}{\lfloor 7}$$

$$= \frac{0.00673 \times 78125}{5040}$$

$$= 0.10432 \qquad \textbf{Ans.}$$

Example 2.33. 15% of a given lot of manufactured goods are defective what is the probability that in a sample of 6 such goods exactly one will be defective?

Solution

Given that

15 % are the detective items

Probability of the detective item is $p = 0.15.$

Probability of the non-defective item $q = 1 - p = 1 - 0.15$

$= 0.85$

For finding the probability that in a sample of 6 such goods exactly one will be defective, Binomial distribution formula can be used

$$b(x;\ n, p) = n_{c_x}\ P^x\ q^{n-x}$$

$$b(1;\ 6,\ 0.15) = 6_{c_1}\ (0.15)^1\ (0.85)^{6-1}$$

$$= 6(0.15)\,(0.44)$$

$$= 0.399$$

Ans.

Example 2.34. Five percent of a given lot of manufactured goods are defective. What is the probability that in a sample of five such goods non will be defective?

Solution

Given that

The probability of the defective item is $p = 0.05$

For finding the probability that in a sample of five such goods non will be defective, Binomial distribution formula can be used

$$b(x;\ n,\ p) = n_{c_x}\ p^x\ q^{n-x}$$

$$b(0;\ 5,\ 0.05) = 5_{c_0}\ (0.05)^0\ (1 - 0{,}05)^{5-0}$$

$$= (0.95)^5$$

$$= 0.773$$

Ans.

Example 2.35. What is the probability that a candidate selected at random will take between 500 to 600 hours to complete the training programme? With a standard deviation of 50 hours.

Solution $$z = \frac{x - \overline{x}}{\sigma}$$

$$z = \frac{600 - 500}{50} = \frac{100}{50} = 0.5$$

The probability from the table for

$z = 0.5$ is 0.1916 **Ans.**

Example 2.36. A firm manufactured plastic bottles of which 2% are defective. The plastic bottles are packed in packets each packet containing 5 bottles. Find the probability of packet which is having no defective bottles.

Solution

$p = 0.02,\ q = 0.98,\ n = 5,\ r = 0$

$$b(r;\ n.p) = \frac{\lfloor n}{\lfloor r\ \lfloor n-r} p^r q^{n-r}$$

$$= \frac{\lfloor 5}{\lfloor 0\ \lfloor 5-0} (0.02)^0 (0.98)^{5-0}$$

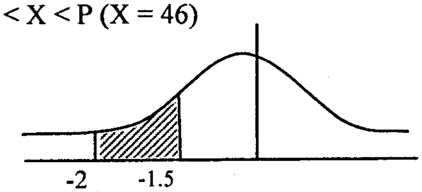
$$= (0.98)^5$$ **Ans.**

Example 2.37. If the mean and S.D. of normal distribution are 70 and 16.

Find P (X = 38) < X < P (X = 46) **(Supple. Nov.2008)**

Solution

P (X = 38) < X < P (X = 46)



$$\Rightarrow P\left(Z_1 = \frac{X-\overline{X}}{\sigma}\right)^n < Z < P\left(Z_2 = \frac{X-\overline{X}}{\sigma}\right)$$

$$\Rightarrow P\left(Z_1 = \frac{38-70}{16}\right)^n < Z < P\left(Z_2 = \frac{46-70}{16}\right)$$

$$\Rightarrow P(Z_1 = -2)^n < Z < P(Z_2 = -1.5)$$

$\Rightarrow$ Area from (0 to -2) - Area from (0 to – 1.5)

$\Rightarrow$ Area from (0 to 2) - Area from (0 to 1.5)

$\Rightarrow$ 0.4772-0.4332

$\Rightarrow$ 0.044 **Ans.**

Example 2.38. A population consists of 5, 10, 14, 18, 13, 24 consider all possible samples of size two which can be drawn without replacement from the population. Find

(Reg. Nov. 2006 Set 2)
(Reg. Nov. 2006 Set 4)
(Supple. Feb. 2007 Set 3)

a. The mean of the population
b. The standard deviation of the population
c. The mean of the sampling distribution of means
d. The standard deviation of sampling distribution of means.

Solution

a. Mean of population

$$\mu = \frac{\sum x_i}{N}$$

$$\mu = \frac{5+10+14+18+13+24}{6}$$

$$= \frac{84}{6}$$

$$= 14 \qquad \textbf{Ans.}$$

b. The standard deviation of the population

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}}$$

$$\sigma = \sqrt{\frac{(5-14)^2+(10-14)^2+(14-14)^2+(18-14)^2+(13-14)^2+(24-14)^2}{6}}$$

$$\sigma = \sqrt{\frac{(-9)^2+(-4)^2+(0)^2+(4)^2+(-1)^2+(10)^2}{6}}$$

$$\sigma = \sqrt{\frac{214}{6}}$$

$$\sigma = \sqrt{35.66}$$

$$= 5.972 \qquad \textbf{Ans.}$$

c. Sampling withoı
without replacen

$$N_{c_n} = 6$$

Here N = Population si
n = Sample size
Listing all possible saı
me get

(5, 10) (10, 14)
(5, 14) (10, 18)
(5, 18) (10, 13)
(5, 13) (10, 24)
(5, 24)

The mean for each of tl

7.5 12 16 15.5
9.5 14 13.5 21
11.5 11.5 19
9 17
14.5

Sampling distribution

Sample mean	$\overline{X}_i$	7.5	9
Freq.	f_i	1	1

Mean of the sampling c

$$\mu_{\overline{X}} = \frac{7.5\times1+9\times1+9.5}{}$$

$$+14.5\times1+$$

$$= \frac{7.5+9+9.5+2}{}$$

$$= \frac{210}{15} = 14$$

d. The standard dev

c. Sampling without replacement finite population the total number of samples without replacement is

$$N_{c_n} = 6_{c_2} = \frac{6}{\angle 2\ \angle 6-2} = \frac{\angle 6}{\angle 2\ \angle 4} = \frac{6\times 5\angle 4}{2\ \angle 4} = 15$$

Here N = Population size

n = Sample size

Listing all possible samples of size 2 from given population without replacement me get

(5, 10)	(10, 14)	(14, 18)	(18, 13)	(13, 24)
(5, 14)	(10, 18)	(14, 13)	(18, 24)	
(5, 18)	(10, 13)	(14, 24)		
(5, 13)	(10, 24)			
(5, 24)				

The mean for each of these samples

7.5	12	16	15.5	18.5
9.5	14	13.5	21	
11.5	11.5	19		
9	17			
14.5				

Sampling distribution of the mean

Sample mean	$\overline{X}_i$	7.5	9	9.5	11.5	12	13.5	14	14.5	15.5	16	17	18.5	19	21
Freq.	f_i	1	1	1	2	1	1	1	1	1	1	1	1	1	1

Mean of the sampling distribution of mean

$$\mu_{\bar{X}} = \frac{7.5\times 1+9\times 1+9.5\times 1+11.5\times 2+12\times 1+13.5\times 1+14\times 1}{15}$$

$$\frac{+14.5\times 1+15.5\times 1+16\times 14+17\times 1+18.5\times 1+19\times 1+21\times 1}{15}$$

$$= \frac{7.5+9+9.5+23+12+13.5+14+14.5+15.5+16+17+18.5+19+21}{15}$$

$$= \frac{210}{15} = 14$$ **Ans.**

d. The standard deviation of sampling distribution of mean

$$\sigma_{\bar{x}}^2 = \frac{\sum_{i=1}^{N} f_i (X_i - \mu)^2}{N}$$

$$\sigma_{\bar{x}}^2 = \frac{1(7.5-14)^2 + 1(9-14)^2 + 1(9.5-14)^2 + 2(11.5-18)^2 + 1(12.14)^2 + 1(13.5-14)^2}{15}$$

$$\frac{+1(14-14)^2 + 1(14.5-14)^2 + 1(15.5-14)^2 + 1(16-14)^2 + 1(17-14)^2 + 1(18.5-14)^2 + 1(19.14)^2}{15}$$

$$\sigma_x^2 = \frac{(6.5)^2 + (-5)^2 + (4.5)^2 + 2(2.5)^2 + (-2)^2 + (-0.5)^2 + (0)^{22}}{15}$$

$$\frac{+(.5)^2 + (1.5)^2 + (2)^2 + (2)^2 + (4.5)^2 + (5)^2}{15}$$

$$= \frac{42.25 + 25 + 20.25 + 12.5 + 4 + 0.25 + 0.25 + 2.25 + 4 + 9 + 20.25 + 25}{15}$$

$$= \frac{165}{15}$$

$$\sigma_x^2 = 11$$

$$\sigma_x = 3.3166$$ **Ans.**

Example 2.39. Samples of size two are taken from the population of 1, 2, 3, 4, 5, 6 without replacement find

(Supple Nov. 2008 Set 4)

e. The mean of the population

f. The standard deviation of the population

g. The mean of the sampling distribution of means

h. The standard deviation of sampling distribution of means.

Solution

e. Mean of population

$$\mu = \frac{\sum x_i}{N}$$

$$\mu = \frac{1+2+3+4+5+6}{6}$$

$$= \frac{21}{6}$$

$$= 3.5$$ **Ans.**

f. The standard deviation of the population

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}}$$

$$\sigma = \sqrt{\frac{(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2}{6}}$$

$$\sigma = \sqrt{\frac{(-2.5)^2 + (-1.5)^2 + (-0.5)^2 + (0.5)^2 + (1.5)^2 + (2.5)^2}{6}}$$

$$\sigma = \sqrt{\frac{17.5}{6}}$$

$$\sigma = \sqrt{2.91}$$

$$= 1.707$$

Ans.

g. Sampling without replacement finite population the total number of samples without replacement is

$$N_{c_n} = 6_{c_2} = \frac{6}{\angle 2 \ \angle 6-2} = \frac{\angle 6}{\angle 2 \ \angle 4} = \frac{6 \times 5 \angle 4}{2 \ \angle 4} = 15$$

Here N = Population size

n = Sample size

Listing all possible samples of size 2 from given population without replacement we get

(1, 2)	(2, 3)	(3, 4)	(4, 5)	(5, 6)
(1, 3)	(2, 4)	(3, 5)	(4, 6)	
(1, 4)	(2, 5)	(3, 6)		
(1, 5)	(2, 6)			
(1, 6)				

The mean for each of these samples

1.5	2.5	3.5	4.5	5.5
2	3	4	5	
2.5	3.5	4.5		
3	4			
3.5				

Sampling distribution of the mean

Sample mean	$\overline{X}_i$	1.5	2	2.5	3	3.5	4	4.5	5	5.5
Freq.	f_i	1	1	2	2	3	2	2	1	1

Mean of the sampling distribution of mean

$$\mu_{\bar{X}} = \frac{1.5\times1+2\times1+2.5\times2+3\times2+3.5\times3+4\times2+4.5\times2+5\times1+5.5\times1}{15}$$

$$= \frac{1.5+2+5+6+10.5+8+9+5+5.5}{15}$$

$$= \frac{52.5}{15} = 3.5 \qquad \textbf{Ans.}$$

h. The standard deviation of sampling distribution of mean

$$\sigma_{\bar{x}}^2 = \frac{\sum_{i=1}^{N} f_i (X_i - \mu)^2}{N}$$

$$\sigma_{\bar{x}}^2 = \frac{1(1.5-3.5)^2 + 1(2-3.5)^2 + 2(2.5-3.5)^2 + 2(3-3.5)^2 + 3(3.5.-3.5)^2}{15}$$

$$\frac{+2(4-3.5)^2 + 2(4.5-3.5)^2 + 1(5-3.5)^2 + 1(5.5-3.5)^2}{15}$$

$$\sigma_x^2 = \frac{(-2)^2 + (-1.5)^2 + 2(1)^2 + 2(-0.5)^2 + 3(0)^2 + 2(0.5)^2 + 2(1)^2 + 1(1.5)^2 + 1(2.0)^2}{15}$$

$$= \frac{4+2.25+2+5.0+0+5.0+2+2.25+4}{15}$$

$$= \frac{26.45}{15}$$

$$\sigma_x^2 = 1.763$$

$$\sigma_x = 1.327 \qquad \textbf{Ans.}$$

Example 2.40. Take 30 slips of paper and label 5 each -4 and 4, four each -3 and 3, three each -2 and 2 and each -1, 0 and 1, if each slip of the paper has the same probability of being drawn find the probabilities of getting -4, -3, -2, -1, 0, 1, 2, 3, 4 and find the mean and variance of this distribution of means.

(Reg. Nov. 2006 Set 3)

(Supple. 2007 Set 2)

Solution

The thirty s

-4, -4, -4, -

-3, -3, -3, -

-2, -2, -2, +

-1, -1, +1, +

0, 0

Probability

Probability

Probability

Probability

Probability

Probability

Probability

Probability

Probability

Probability

X
p(X = x

Mean $\mu_{\bar{X}} = \sum X$

$= \sum ($

$+$

$= 0$

Solution

The thirty slips are

-4, -4, -4, -4, -4, +4, +4, +4, +4, +4

-3, -3, -3, -3, +3, +3, +3, +3

-2, -2, -2, +2, +2, +2

-1, -1, +1, +1

0, 0

Probability of getting $-4 = \frac{5}{30} = \frac{1}{6}$

Probability of getting $-3 = \frac{4}{30} = \frac{2}{15}$

Probability of getting $-2 = \frac{3}{30} = \frac{1}{10}$

Probability of getting $-1 = \frac{2}{30} = \frac{1}{15}$

Probability of getting $0 = \frac{2}{30} = \frac{1}{15}$

Probability of getting $1 = \frac{2}{30} = \frac{1}{15}$

Probability of getting $2 = \frac{3}{30} = \frac{1}{10}$

Probability of getting $3 = \frac{4}{30} = \frac{2}{15}$

Probability of getting $4 = \frac{5}{30} = \frac{1}{6}$

Probability distribution of the sample

X	-4	-3	-2	-1	0	1	2	3	4
p(X = x)	$\frac{1}{6}$	$\frac{2}{15}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{15}$	$\frac{1}{6}$

Mean $\mu_{\bar{X}} = \sum X\, p(X)$

$$= \sum \; (-4\times\frac{1}{6}) + (-3\times\frac{2}{15}) + (-2\times\frac{1}{10}) + (-1\times\frac{1}{15}) + (0\times\frac{1}{15}) + (1\times\frac{1}{15})$$

$$+ (2\times\frac{1}{10}) + (3\times\frac{2}{15}) + (4\times\frac{1}{6})$$

$$= 0$$

Variance $\sigma_{\bar{x}}^2 = \sum (X - \mu_{\bar{X}})^2 p(X)$

$$= \sum \ (-4-0)^2 \frac{1}{6} + (-3-0)^2 \frac{2}{15} + (-2-0)^2 \times \frac{1}{10} + (-1-0)^2 \times \frac{1}{15}$$

$$+ (0-0)^2 \times \frac{1}{15} + (1-0)^2 \times \frac{1}{15} + (2-0)^2 \times \frac{1}{10} + (3-0)^2 \times \frac{2}{15}$$

$$+ (4-0)^2 \times \frac{1}{6}$$

$$= 16 \times \frac{1}{6} + 9 \times \frac{2}{15} + 4 \times \frac{1}{10} + 1 \times \frac{1}{15} + 1 \times \frac{1}{15} + 4 \times \frac{1}{10} + 9 \times \frac{2}{15} + 16 \times \frac{1}{16}$$

$= 2.666 + 1.2 + 0.4 + 0.060 + 0.066 + 0.4 + 1.2 + 2.666$

$= 8.664$ **Ans.**

Example 2.41. If the population is 3, 6, 9, 15. 27

a. List all possible samples of size 3 that can be taken without replacement from the finite population.

b. Calculate the mean of each of the sampling distribution of means

c. Find the standard deviation of sampling distribution of means.

(Supple. Nov./Dec. 2000 Set 2)
(Supple. Feb. 2007 Set 1)
(Supple. Nov 2008.Set.2, 3)
(Nov.2009 set 4)

Solution

Given population is 3, 6, 9, 15, and 27

a. List all possible samples of size 3 that can be taken without replacement from the finite population i.e.

$$5_{c_3} = \frac{\angle 5}{\angle 3 \ \angle 5-3}$$

$$= \frac{\angle 5}{\angle 3 \ \angle 2}$$

$$= \frac{5 \times 4 \times \angle 3}{\angle 3 \times 2 \times 1}$$

$$= 10$$

Total number of samples 10.

(3, 6, 9) (3, 6, 15) (3, 6, 27) (6, 9, 15)

(6, 9, 27)
(9, 15, 27)

b. Calculate th
Mean of the

1. (3, 6, 9) me

2. (3; 6, 15) m

3. (3, 6, 27) m

4. (6, 9, 15) m

5. (6, 9, 27) m

6. (3, 9, 15) m

7. (3, 9, 27) m

8. (9, 15, 27) n

9. (6, 15, 27) n

10. (3, 15, 27) n

Sampling distribu

Sample mea
Frequency f

Mean of the sampl

$(6 \times 1) + (8$

$\mu_{\bar{x}} = $

(6, 9, 27) (3, 9, 15) (3, 9, 27)
(9, 15, 27) (6, 15, 27) (3, 15, 27)

Ans.

b. Calculate the mean of each of the sampling distribution of means

Mean of the samples

1. (3, 6, 9) mean $= \dfrac{3+6+9}{3} = \dfrac{18}{3} = 6$

2. (3, 6, 15) mean $= \dfrac{3+6+15}{3} = \dfrac{24}{3} = 8$

3. (3, 6, 27) mean $= \dfrac{3+6+27}{3} = \dfrac{36}{3} = 12$

4. (6, 9, 15) mean $= \dfrac{6+9+15}{3} = \dfrac{30}{3} = 10$

5. (6, 9, 27) mean $= \dfrac{6+9+27}{3} = \dfrac{42}{3} = 14$

6. (3, 9, 15) mean $= \dfrac{3+9+15}{3} = \dfrac{27}{3} = 9$

7. (3, 9, 27) mean $= \dfrac{3+9+27}{3} = \dfrac{39}{3} = 13$

8. (9, 15, 27) mean $= \dfrac{9+15+27}{3} = \dfrac{51}{3} = 17$

9. (6, 15, 27) mean $= \dfrac{6+15+27}{3} = \dfrac{48}{3} = 16$

10. (3, 15, 27) mean $= \dfrac{3+15+27}{3} = \dfrac{45}{3} = 15$

Sampling distribution of the mean

Sample mean $\overline{X}_i$	6	8	9	10	12	13	14	15	16	17
Frequency f_i	1	1	1	1	1	1	1	1	1	1

Mean of the sampling distribution of mean

$$\mu_{\bar{x}} = \frac{(6\times 1)+(8\times 1)+(9\times 1)+(10\times 1)+(12\times 1)+(13\times 1)+(14\times 1)+(15\times 1)+(16\times 1)+(17\times 1)}{10}$$

$$= \frac{6+8+9+10+12+13+14+15+16+17}{10}$$

$$= \frac{120}{10} = 12 \qquad \textbf{Ans.}$$

C. Standard deviation of sampling distribution of means

Variance

$$\sigma_{\bar{x}}^2 = \frac{\sum_{i=1}^{N} f_i (X_i - \mu)^2}{N}$$

$$\sigma_{\bar{X}}^2 = \frac{1(6-12)^2 + 1(8-12)^2 + 1(9-12)^2 + 1(10-12)^2 + 1(12-12)^2 + 1(13-12)^2 + 1(14-12)^2 + 1(15-12)^2 + 1(16-12)^2 + 1(17-12)^2}{10}$$

$$= \frac{(-6)^2 + (-4)^2 + (-3)^2 + (-2)^2 + (0)^2 + (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2}{10}$$

Variance $\sigma_{\bar{X}}^2 = \dfrac{120}{10} = 12$

S.D. $\sigma_{\bar{X}} = 3.464$ **Ans.**

Example 2.42. A population consists of five numbers 2, 3, 6, 8, 11. Consider all possible samples of size two which can be drawn without replacement from the population. Find

(NR/OR June 2002)
(Supple. Nov. /Dec. 2004)
(Reg. April/May 2005 Set 3 Set 4)
(Supple. Nov 2008.Set.1)

a. The mean of the population
b. Standard deviation of the population
c. The mean of the sampling distribution of means
d. The standard deviation of the sampling distribution of means.

Solution

a. The mean of the population

$$\mu = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6 \qquad \textbf{Ans.}$$

b. Standard d

$\sigma = \sqrt{\frac{\sum_{i=1}^{N}}{}}$

$= \sqrt{(2-}$

$= \sqrt{16+}$

$= 3.3$

c. Sampling w
without rep

Here N = pc
n = sample s
Listing
replacement

(2, 3) (3,
(2, 6) (3,
(2, 8) (3,
(2, 11)

The mean of each c

2.5 4.5 7
4 5.5
5 7
6.5

Sample n
Frequenc

The mean of the san

$\mu_{\bar{X}} = \frac{2.5 \times 1 + 4 \times}{}$

b. Standard deviation of population

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}}$$

$$= \sqrt{\frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (8-6)^2 + (11-6)^2}{5}}$$

$$= \sqrt{\frac{16+9+0+4+25}{5}} = \sqrt{\frac{54}{5}} = \sqrt{10.8} = 3.28$$

$= 3.3$ **Ans.**

c. Sampling without replacement (finite population, the total number of samples without replacement is

$$N_{C_n} = 5_{C_2} = \frac{\angle 5}{\angle 2\ \angle 5-2} = \frac{5 \times 4\ \angle 3}{2\ \angle 3} = 10$$

Here N = population size

n = sample size

Listing all possible samples of size 2 from given population without replacement we get

(2, 3) (3, 6) (6, 8) (8, 11)
(2, 6) (3, 8) (6, 11)
(2, 8) (3, 11)
(2, 11)

The mean of each of these samples

2.5 4.5 7 9.5
4 5.5 8.5
5 7
6.5

Sample mean $\overline{X}_i$	2.5	4	4.5	5	5.5	6.5	7	8.5	9.5
Frequency f_i	1	1	1	1	1	1	1	1	1

The mean of the sampling distribution of mean

$$\mu_{\overline{X}} = \frac{2.5\times1 + 4\times1 + 4.5\times1 + 5\times1 + 5.5\times1 + 6.5\times1 + 7\times2 + 8.5\times1 + 9.5\times1}{10}$$

$$= \frac{2.5+4+4.5+5+5.5+6.5+14+8.5+9.5}{10}$$

$$= \frac{60}{10} = 6$$ **Ans.**

d. The standard deviation of sampling distribution of mean

$$\sigma_{\bar{x}}^2 = \frac{\sum_{i=1}^{N} f_i (X_i - \mu)^2}{N}$$

$$\sigma_{\bar{x}}^2 = \frac{(-3.5)^2 + (-2)^2 + (-1.5)^2 + (-1)^2 + (-0.5)^2 + (0.5)^2 + 2\times(1)^2 + (2.5)^2 + (3.5}{10}$$

$$= \frac{12.25+4+2.25+1+0.25+0.25+2+6.25+12.25}{10}$$

$$= \frac{40.5}{10}$$

$$\sigma_{\bar{X}}^2 = 4.05$$

$$\sigma_{\bar{X}} = 2.01$$ **Ans.**

Example 2.43. Find the value of the finite population correction factor for n = 10 and N = 1000

(JNTU 1999), (2000/S)

Solution

N → population size

n → sample size

Given that

N = 1000

n = 10

Correction factor for finite population $= \frac{N-n}{N-1}$

Correction factor $= \frac{1000-10}{1000-1} = \frac{990}{999} = 0.99$ **Ans.**

Example 2.44. What is the probability that X will be between 75 and 78 if a random sample of size 100 taken from an infinite population has mean 76 and variance 256.

(Reg. Nov. 2006 Set 1)

(Supple. Nov. 2004 Set 3)

(NR / OR 2001)

Solution

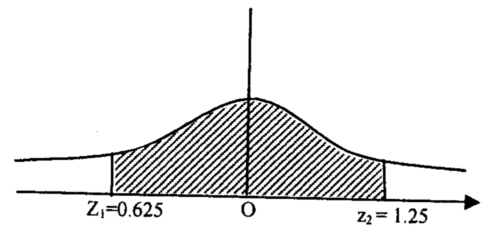
Given that

Sample size n = 100

Mean of population = 76

Variance of population $\sigma^2 = 256$

$\sigma = 16$



$\overline{X}_1 = 75$ $\qquad$ $\overline{X}_2 = 78$

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \qquad z_1 = \frac{75 - 76}{16/\sqrt{100}} \qquad z_2 = \frac{78 - 76}{16/\sqrt{100}}$$

$z_1 = -0.0625$ $\qquad$ $z_2 = 1.25$

$p(75 \le x \le 78) = p(-0.625 \le z \le 1.25)$

= Area (from 0 to -0.625) + Area (from 0 to 1.25)

= 0.2340 + 0.3944

= 0.6284 **Ans.**

Example 2.45. A population consists of six numbers 4, 8, 12, 16, 20, 24 consider all samples of size two which can be drawn without replacement from this population. Find

(JNTU 2000)
(Supple. Feb. 2007 Set 4)

1. The population mean
2. The population standard deviation
3. The mean of the sampling distribution of means
4. The standard deviation of the sampling distribution of means.

Verify (3) & (4) from (1) & (2) by the use of suitable formulae.

Solution

1. Mean of population $\mu = \dfrac{\sum x_i}{N}$

$$= \frac{4 + 8 + 12 + 16 + 20 + 24}{6}$$

$$= \frac{84}{6}$$

$= 14$ **Ans.**

2. Population standard deviation

$$\sigma = \sqrt{\frac{\sum (x_i - \mu)^2}{N}}$$

$$= \sqrt{\frac{(4-14)^2 + (8-14)^2 + (12-14)^2 + (16-14)^2 + (20-14)^2 + (24-14)^2}{6}}$$

$$= \sqrt{\frac{100+36+4+4+36+100}{6}}$$

$$= \sqrt{\frac{280}{6}} \quad = \sqrt{46.67}$$

$\sigma = 6.8313$ **Ans.**

3. Sampling without replacement (finite population)

The total number of samples without replacement is

$$N_{C_n} = 6_{C_2} = \frac{\angle 6}{\angle 2 \ \angle 6-2} = \frac{6 \times 5 \times \angle 4}{2 \times \angle 4} = 15$$

Here N = population size, n = sample size

Listing all possible samples of size 2 from given population without replacement we get

(4, 8)	(8, 12)	(12, 16)	(16, 20)	(20, 24)
(4, 12)	(8, 16)	(12, 20)	(16, 24)	
(4, 16)	(8, 20)	(12, 24)		
(4, 20)	(8, 24)			
(4, 24)				

The mean for each of these samples:

6	10	14	18	22
8	12	16	20	
10	14	18		
12	16			
14				

Sample mean $\overline{X}_i$	6	8	10	12	14	16	18	20	22
Frequency f_i	1	1	2	2	3	2	2	1	1

Mean of the sampling distribution of mean

$$\mu_{\overline{X}} = \frac{6\times1+8\times1+10\times2+12\times2+14\times3+16\times2+18\times2+20\times1+22\times1}{15}$$

$$= \frac{210}{15}$$

$$= 14 \qquad \textbf{Ans.}$$

4. The standard deviation of the sampling distribution of mean

$$\sigma_{\bar{x}}^2 = \frac{\sum_{i=1}^{N} f_i (X_i - \mu)^2}{N}$$

$$\sigma_{\bar{X}}^2 = \frac{1(6-14)^2 + 1(8-14)^2 + 2(10-14)^2 + 2(12-14)^2 + 3(14-14)^2}{15}$$

$$\frac{+2(16-14)^2 + 2(18-14)^2 + 1(20-14)^2 + 1(22-14)^2}{15}$$

$$= \frac{(-8)^2 + (-6)^2 + 2(-4)^2 + 2(-2)^2 + 3(0)^2 + 2(2)^2 + 2(4)^2 + (6)^2 + (8)^2}{15}$$

$$= \frac{64 + 36 + 32 + 8 + 8 + 32 + 36 + 64}{15}$$

$$= \frac{280}{15}$$

$$\sigma_{\bar{X}}^2 = 18.666$$

$$\sigma_{\bar{X}} = 4.3204 \qquad \textbf{Ans.}$$

Varification

$$\mu = \mu_{\bar{X}} = 14$$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{N}} \sqrt{\frac{N-n}{N-1}}$$

$$= \frac{6.831}{2.449} \times 0.8944$$

$$\sigma_{\bar{X}} = 2.494 \qquad \textbf{Ans.}$$

Example 2.46. A normal population has a standard deviation of a sample of size 400 is collected for testing, find the standard error of the mean of sampling distribution.

Solution

Given that

Standard deviation of the population $\sigma = 2$

Sample size n = 400

Standard error S.E. = $\dfrac{\sigma}{\sqrt{n}} = \dfrac{2}{\sqrt{400}} = \dfrac{2}{20} = \dfrac{1}{10} = 0.01$ **Ans.**

Example 2.47. Find the value of correction factor if n = 4 and N = 100

Solution

Given that

The size of population N = 100

Size of sample n = 4

Correction factor = $\dfrac{N-n}{N-1} = \dfrac{100-4}{100-1} = \dfrac{96}{99} = 0.9696$ **Ans.**

Example 2.48. How many different samples of size 2 can be chosen, from a finite population of size 50?

Solution

Given that

The size of population N = 50

Size of sample n = 2

The total number of samples without replacement (infinite population) is

$$N_{C_n} = 50_{C_2} = \frac{\angle 50}{\angle 2\ \angle 50-2} = \frac{50 \times 49 \times \angle 48}{2 \times \angle 48} = 1225$$ **Ans.**

Example 2.49. Determine the mean and s.d. of the sampling distribution of means of 300 random samples each of size n =36 are drawn from a population of N = 1400 which is normally distributed with mean $\mu = 21.5$ and s.d. $\sigma = 0.058$, if sampling is done

a. with replacement and b. without replacement

Solution

Given that

Size of population N = 1400

Size of sample n = 36

Mean $\mu = 21.5$

s.d. $\sigma = 0.058$

a. With replacement (infinite population)

Mean of the sampling distribution of means

$$\mu_{\bar{X}} = \mu = 21.5$$

Standard deviation of the sampling distribution of means

$$\mu_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{0.058}{\sqrt{36}} = .00966 \qquad \textbf{Ans.}$$

b. Without replacement

$$\mu_{\bar{X}} = \mu = 21.5$$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \frac{0.058}{\sqrt{36}}\sqrt{\frac{1400-36}{1400-1}}$$

$$= 0.0096\sqrt{\frac{1364}{1399}}$$

$$= 0.0096 \times 0.9874$$

$$= 0.009479 \qquad \textbf{Ans.}$$

Example 2.50. Determine the expected number of samples whose mean

a. Lies between 66.8 and 69.8

b. is less than 67.4

If normal population mean is 68 and standard deviation is 3 and number of samples 70.

Solution Given that

population mean $\mu = 68$

population standard diviation $\sigma = 3$

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$z_1 = \frac{66.8 - 68.0}{3/\sqrt{25}} = \frac{-1.2}{0.6} = -2 \qquad z_2 = \frac{69.8 - 68.0}{3/\sqrt{25}} = \frac{1.8}{0.6} = 3$$

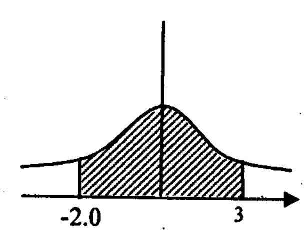
a. Lies between 66.8 and 69.8

$$p(66.8 < \bar{X} < 69.8) = p(-2.0 < z < 3)$$

= Area (from 0 to -2.0) + Area (from 0 to 3)

= 0.4772 + 0.4987

= 0.9759



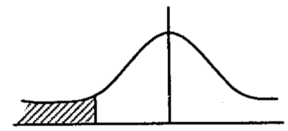
Expected number of samples with mean between 66.8 and 69.8 is

$= \text{number of samples} \times \text{probability}$

$= 70 \times 0.9759$

$= 68.313$ **Ans.**

b. is less than 65.4 $z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = \dfrac{67.4 - 68.0}{3/\sqrt{25}}$



$= \dfrac{-0.6}{0.6} = -1$

$p(\overline{X} < 65.4) = p(z < -1.0)$

$= 0.5 - 0.3413$

$= 0.1587$

Expected number of samples = number of samples × probability

$= 70 \times 0.1587$

$= 11.109$ **Ans.**

Example 2.51. Let $U_1 = (1,3.5), U_2 = (3,8)$ find

a. μ_{U_1} b. μ_{U_2} c. $\mu_{U_1+U_2}$ d. $\mu_{U_1-U_2}$

e. σ_{U_1} f. σ_{U_2} g. $\sigma_{U_1+U_2}$ h. $\sigma_{U_1-U_2}$

Verify that

i. $\mu_{U_1+U_2} = \mu_{U_1} + /_{2}$

j. $\mu_{U_1-U_2} = \mu_{U_1} - \mu_{U_2}$

k. $\sigma_{U_1 \pm U_2} = \sqrt{\sigma_{U_1}^2 + \sigma_{U_2}^2}$

Solution

Given that $U_1 = (1,3,5), U_2 = (3,8)$

a. $\mu_{U_1} = \dfrac{1+3+5}{3} = \dfrac{9}{3} = 3$ **Ans.**

b. $\mu_{U_2} = \dfrac{3+8}{2} = \dfrac{11}{2} = 5.5$ **Ans.**

c. $\mu_{U_1+U_2} = \mu_{U_1} + \mu_{U_2}$

$$= 3 + 5.5$$

$$= 8.5$$ **Ans.**

d. $\mu_{U_1-U_2} = \mu_{U_1} - \mu_{U_2}$

$$= 3 - 5.5$$

$$= -2.5$$ **Ans.**

e. $\sigma_{U_1} = ?$

$\sigma^2_{U_1}$ = Variance of the population U_1

$$= \frac{(1-3)^2 + (3-3)^2 + (5-3)^2}{3}$$

$$= \frac{(-2)^2 + (0)^2 + (2)^2}{3}$$

$$= \frac{4+4}{3}$$

$$= \frac{8}{3}$$

$$\sigma^2_{U_1} = 2.666$$

$$\sigma_{U_1} = 1.6329$$ **Ans.**

f. $\sigma_{U_2} = ?$

${\sigma_{U_2}}^2$ = Variance of the population U_2

$$= \frac{(3-5.5)^2 + (8-5.5)^2}{2}$$

$$= \frac{(-2.5)^2 + (2.5)^2}{2}$$

$$= \frac{6.25 + 6.25}{2}$$

$$= \frac{12.5}{2}$$

$$\sigma_{U_2}{}^2 = 6.25$$

$$\sigma_{U_2} = 2.5 \qquad \textbf{Ans.}$$

g. $$\sigma_{U_1+U_2} = \sqrt{\sigma^2_{U_1} + \sigma^2_{U_2}}$$

$$= \sqrt{(3.66) + 6.25}$$

$$= \sqrt{8.916}$$

$$\sigma_{U_1+U_2} = 2.985 \qquad \textbf{Ans.}$$

h. $$\sigma_{U_1-U_2} = \sqrt{\sigma^2_{U_1} - \sigma^2_{U_2}}$$

$$= \sqrt{2.666 + 6.25}$$

$$= \sqrt{8.916}$$

$$= 2.985 \qquad \textbf{Ans.}$$

i. Verification

$$\sigma_{U_1+U_2} = \mu_{U_1} + \mu_{U_2} = 3 + 5.5 = 8.5$$

$$\mu_{U_1+U_2} = ?$$

$$U_1 = (1,3,5)\ U_2 = (3,8)$$

$U_1 + U_2$ = {1+3, 3+3, 5+3, 1+8, 3+8, 5+8}

= {4, 6, 8, 9, 11, and 13}

$$\mu_{U_1+U_2} = \frac{4+6+8+9+11+13}{6}$$

$$= \frac{51}{6}$$

$$= 8.5 = \mu_{U_1} + \mu_{U_2} \qquad \textbf{Ans.}$$

j. $$\mu_{U_1-U_2} = \mu_{U_1} - \mu_{U_2}$$

$$U_1 = (1,3,5), U_2 = (3,8)$$

$$U_1 - U_2 = \{1-3,\ 3-3, 5-3, 1-8, 3-8, 5-8\}$$

= {-2, 0, 2, -7, -5, -3}

$$\mu_{U_1-U_2} = \frac{(-2)+0+2+(-7)+(-5)+(-3)}{6}$$

k. $\sigma_{U_1 \pm U_2} = \sqrt{}$

$U_1 + U_2 =$

$\sigma^2_{U_1+U_2} = \frac{(4-8.5}{}$

$= \frac{(4.5)^2 +}{}$

$= \frac{20.25 +}{}$

$= \frac{53.5}{6}$

$\sigma^2_{U_1+U_2} = 8.$

$\sigma_{U_1+U_2} = 2.9$

$U_1 - U_2 = \{$

$\sigma^2_{U_1-U_2} =$

$[-2-(-2.5)]^2[-0-(-$

$= \frac{(-2+2.5)^2(2.}{}$

$= \frac{(0.5)^2 + (2.5)^2}{}$

$= \frac{0.2}{}$

$= \frac{53.}{6}$

$\sigma^2_{U_1-U_2}$

$\sigma_{U_1-U_2} = 2.9$

$$= \frac{-15}{6}$$

$$= -2.5 = \mu_{U_1} - \mu_{U_2} \qquad \textbf{Ans.}$$

k. $$\sigma_{U_1 \pm U_2} = \sqrt{\sigma^2{}_{U_1} + \sigma^2{}_{U_2}}$$

$$U_1 + U_2 = \{ 4, 6, 8, 9, 11, 13\}$$

$$\sigma^2_{U_1+U_2} = \frac{(4-8.5)^2 + (6-8.5)^2 + (8-8.5)^2 + (9-8.5)^2 + (11-8.5)^2 + (13.8.5)^2}{6}$$

$$= \frac{(4.5)^2 + (2.5)^2 + (0.5)^2 + (0.5)^2 + (2.5)^2 + (4.5)^2}{6}$$

$$= \frac{20.25 + 6.25 + 0.25 + 0.25 + 6.25 + 20.25}{6}$$

$$= \frac{53.5}{6}$$

$$\sigma^2_{U_1+U_2} = 8.9166$$

$$\sigma_{U_1+U_2} = 2.986 = \sqrt{\sigma^2_{U_1} + \sigma^2_{U_2}}$$

$$U_1 - U_2 = \{-2, 0, 2, -7, -5, -3\}$$

$$\sigma^2_{U_1-U_2} =$$

$$\frac{[-2-(-2.5)]^2[-0-(-2.5)]^2 + [2-(-2.5)]^2 + [-7-(-2.5)]^2 + [-5-(-2.5)]^2 + [-3-(-2.5)]^2}{6}$$

$$= \frac{(-2+2.5)^2(2.5)^2 + (2+2.5)^2 + (-7+2.5)^2 + (-5+2.5)^2 + (-3+2.5)^2}{6}$$

$$= \frac{(0.5)^2 + (2.5)^2 + (4.5)^2 + (-4.5)^2 + (-4.5)^2 + (-2.5)^2 + (-0.5)^2}{6}$$

$$= \frac{0.25 + 6.25 + 20.25 + 20.25 + 6.25 + 0.25}{6}$$

$$= \frac{53.5}{6}$$

$$\sigma^2_{U_1-U_2} = 8.9166$$

$$\sigma_{U_1-U_2} = 2.9860 = \sqrt{\sigma^2_{U_1} + \sigma^2_{U_2}} \qquad \textbf{Ans.}$$

Example 2.52. Explain normal distribution. Find the probability that a random variable having the standard normal distribution will take on a value between 0.87 and 1.28.

(NR / OR 2001)

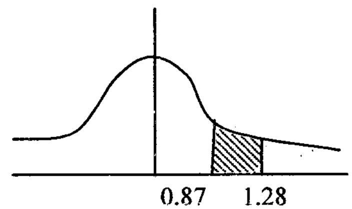
Solution

$P(0.87 \le z \le 1.28)$

= Area from (0 to 1.28) – Area (from 0 to 0.87)

$= 0.3997 - 0.3078$

$= 0.0919$ **Ans.**

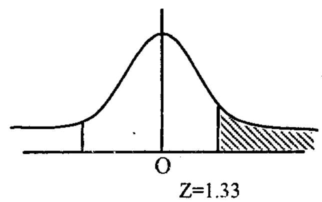


Example 2.53. If the masses of 300 students are normally distributed with mean 68kg. and standard deviation 3 kg., how many students have masses.

i. Greater than 72kg

ii. Less than or equal to 64kg

iii. Between 65 and 71kg (inclusive)



Solution

Given that

The masses of 300 students are normally distributed with mean

$\mu = 68$ kgms.

Standard deviation $\sigma = 3$ kgms.

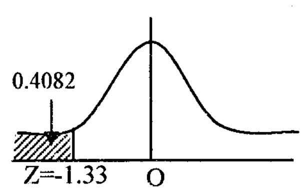
i. Greater than 72 kgms.

$$z = \frac{72-68}{3} = 1.33$$

$$p(X > 72) = 0.5\text{-(Area } z = 1.33)$$
$$= 0.5 - 0.4082$$
$$= 0.5 - 0.4082$$
$$= 0.0918$$

No. of students whose weight > 72

$$= 0.0918 \times 300$$
$$= 28$$



ii. Less than or equal to 64 $p(X \le 64)$

$$z = \frac{64-68}{3} = 1.33$$

$P(x \le 64) = 0.5 -$ Area $z = -1.33$

No. of stud

iii. Between 65

P (65 < x <

$z_1 = \frac{65 - (}{3}$

= - 1

P(65 > X <

= Are

= 0.34

= 0.68

The number

= 300

Example 2.54. De
sample of 40 of 1 l
1 liter of such pai
31.5 square feet.

Solution

Given that

Sample of 40 of 1 l

Between 510 to 52

Mean = 513.3 squa

Standard deviation

We know that

$= 0.5 - 0.4082$

$= .0918$

No. of students whose weight is less or equal to 64 kgms

$= 0.918 \times 300$

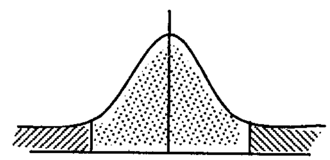
$= 28$

iii. Between 65 and 71 k.gms. (inclusive)

P (65 < x < 71)

$$z_1 = \frac{65-68}{3} \qquad z_2 = \frac{71-68}{3}$$

$$= -1 \qquad\qquad = +1$$



$P(65 > X < 71) = p(-1 < z < 1)$

= Area 0 to -1 + area 0 to +1

$= 0.3413 + 0.3413$

$= 0.6826$

The number of students whose masses lie between 65 and 71 k.gms. is

$= 300 \times 0.6826 \quad = 205$ **Ans.**

Example 2.54. Determine the probability that the sample mean area covered by a sample of 40 of 1 liter paint boxes will be between 510 to 520 square feet given that 1 liter of such paint box covers on the average 513.3 square feet with standard of 31.5 square feet. **(Reg. 2004 April/May Set 3)**

Solution

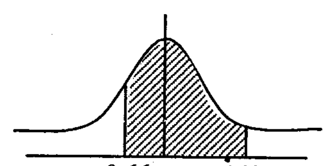
Given that

Sample of 40 of 1 liter paint boxes n = 40

Between 510 to 520 square feet

Mean = 513.3 square feet

Standard deviation = 31.5 square feet



We know that

$$z = \frac{X - \overline{X}}{\sigma / \sqrt{n}}$$

$$z_1 = \frac{510 - 513.3}{31.5/\sqrt{40}} \qquad z_2 = \frac{520 - 513.3}{31.5/\sqrt{40}}$$

$$z_1 = \frac{-3.3}{4.9806} \qquad z_2 = \frac{6.7}{4.9806}$$

$= -0.66257 \; z_2 = 1.3452$

$$p(x_1 \le \bar{x} \le x_2) = p(510 \le \bar{x} < 520)$$

$$= p(z_1 \le z \le z_2) = p(-0.66 \le z \le 1.3452)$$

$$= \text{area (from 0 to} -0.66) + (\text{area from o to } 1.3452)$$

$$= 0.2454 + 0.4099$$

$$= 0.6553 \qquad \textbf{Ans.}$$

Example 2.55.

A population random variable X has mean 100 and standard deviation 16. What are the mean and standard deviation of the sample mean for random samples of size 4 drawn with replacement. **(Reg. April / May 2004 Set 1)**

Solution

Given that

Mean $\mu = 100$, Standard deviation $\sigma = 16$, n = 4

Mean & standard deviation of the sample mean =?

With replacement

$\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \sigma / \sqrt{n}$

$$\mu_{\bar{X}} = \mu = 100 \text{ and } \sigma_{\bar{x}} = \frac{16}{\sqrt{4}} = \frac{16}{2} = 8$$

Ans. $\boxed{\mu_{\bar{X}} = 100}$ **Ans.** $\boxed{\sigma_{\bar{X}} = 8}$

Example 2.56.

Let S = {1, 5, 6, 8} find the probability distribution of the sample mean for random sample of size 2 drawn without replacement.

(Reg. April / May 2004 Set 1)

Solution

The total number of samples without replacement is

$$N_{C_n} = 4_{C_2} = \frac{\angle 4}{\angle 2\ \angle 4-2}$$

$$= \frac{\angle 4}{\angle 2 \angle 2}$$

$$= \frac{4 \times 3 \times \angle 2}{2\ \angle 2}$$

$$= 6$$

The number of

(1,

Mean of the ea

$(1,5)$ *mean*

$(1,6)$ *mean*

$(1,8)$ *mean*

$(5,6)$ *mean*

$(5,8)$ *mean*

$(6,8)$ *mean*

Mean of sample

Sampling distrit

Population

Example 2.57. having the mean 72 and 77?

Solution

Given that

n = 144

The number of samples = 6

(1, 5) (1, 6) (1, 8) (5, 6) (5, 8) (6, 8)

Mean of the each sample

$$(1,5) \quad mean \; = \frac{1+5}{2} = \frac{6}{2} = 3$$

$$(1,6) \quad mean \; = \frac{1+6}{2} = \frac{7}{3} = 3.5$$

$$(1,8) \quad mean \; = \frac{1+8}{2} = \frac{9}{2} = 4.5$$

$$(5,6) \quad mean \; = \frac{5+6}{2} = \frac{11}{2} = 5.5$$

$$(5,8) \quad mean \; = \frac{5+8}{2} = \frac{13}{2} = 6.5$$

$$(6,8) \quad mean \; = \frac{6+8}{2} = \frac{14}{2} = 7$$

Mean of samples

= 3, 3.5, 4.5, 5.5, 6.2, 7

Sampling distribution mean (without replacement)

$\overline{X}_i$:	3	3.5	4.5	5.5	6.5	7
f_i :	1	1	1	1	1	1

$$\text{Population mean} = \frac{1+5+6+8}{4} = \frac{20}{4} = 5$$

$$\mu_{\overline{X}} = \frac{3\times +3.5\times 1+4.5\times 1+5.5\times 1+6.5\times 1+7\times 1}{6}$$

$$= \frac{30}{6} = 5$$

Ans.

Example 2.57. A random sample of size 144 is taken from an infinite population having the mean 75 and variance 225. What is the probability that $\overline{x}$ will be between 72 and 77? **(Supple. Nov. /Dec. 2005 Set 3)**

Solution

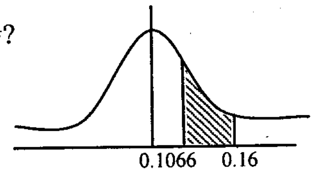
Given that

n = 144

Mean $\overline{X} = 75$

Variance $\sigma = 225$

The probability that $\overline{x}$ will be between 72 and 77 =?



We have

$$z = \frac{X - \overline{X}}{\sigma / \sqrt{n}}$$

$$z_1 = \frac{72 - 75}{225 / \sqrt{144}}, \qquad z_2 = \frac{77 - 75}{225 / \sqrt{144}}$$

$$z_1 = \frac{-3}{18.75} \qquad z_2 = \frac{2}{18.75}$$

$$z_1 = 0.16 \qquad z_2 = 0.1066$$

$$p(x_1 \le X \le x_2) = p(72 \le X \le 77)$$

$$= p(z_1 \le z \le z_2)$$

$$= p\,(0.16 \le z \le 0.1066)$$

= Area (from 0 to 0.16) - Area from (0 to 0.1066)

$$= 0.0636 - 0.0398$$

$$= 0.0238$$

Ans.

Example 2.58. Find the mean and standard deviation of sampling distribution of variances for the population 3, 4, 5, 6 by drawing samples of size two with replacement.

Solution

The total number of samples with replacement is

$N^n = 4^2 = 16$

Here N = population size and n = sample size

Total number of samples = 16

(3, 3)	(4, 3)	(5, 3)	(6, 3)
3	3.5	4	4.5
(3, 4)	(4, 4)	(5, 4)	(6, 4)
3.5	4	4.5	5
(3, 5)	(4, 5)	(5, 5)	(6, 5)
4	4.5	5	5.5
(3, 6)	(4, 6)	(5, 6)	(6, 6)
4.5	5	5.5	6

Variance of each

Variance for the :

Variance for the :

$= \frac{(4 - 3.5}{}$

Variance for the :

$= \frac{(5-4)^2}{}$

Variance for the :

$= \frac{(6 - 4.5}{}$

Variance for the s

$= \frac{(3 - 3.5}{}$

Variance for the s

$= \frac{(4-4)^2}{}$

Variance for the s

$= \frac{(5 - 4.5)}{}$

Variance for the s

Variance for the s

Variance for the s

$= \frac{(4 - 4.5)}{}$

0.16

Ans.

listribution of
ize two with

Variance of each sample

Variance for the sample (3, 3) with mean 3

$$= \frac{(3-3)^2 + (3.3)^2}{2} = 0$$

Variance for the sample (4, 3) with mean 3.5

$$= \frac{(4-3.5)^2 + (3-3.5)^2}{2} = \frac{(0.5)^2 + (-0.5)^2}{2} = \frac{0.25+0.25}{2} = 0.25$$

Variance for the sample (5, 3) with mean 4

$$= \frac{(5-4)^2 + (5-4)^2}{2} = \frac{(1)^2 + (1)^2}{2} = \frac{2}{2} = 1$$

Variance for the sample (6, 3) with mean 4.5

$$= \frac{(6-4.5)^2 + (3-4.5)^2}{2} = \frac{(1.5)^2 + (-1.5)^2}{2} = \frac{2.25+2.25}{2} = 2.25$$

Variance for the sample (3, 4) with mean 3.5

$$= \frac{(3-3.5)^2 + (4-3.5)^2}{2} = \frac{(-0.5)^2 + (0.5)^2}{2} = \frac{0.25+0.25}{2} = 0.25$$

Variance for the sample 4, 4) with mean 4

$$= \frac{(4-4)^2 + (4-4)^2}{2} = 0$$

Variance for the sample (5, 4) with mean 4.5

$$= \frac{(5-4.5)^2 + (4-4.5)^2}{2} = \frac{(0.5)^2 + (-0.5)^2}{2} = \frac{0.25+0.25}{2} = 0.25$$

Variance for the sample (6, 4) with mean 5

$$= \frac{(6-5)^2 + (4-5)^2}{2} = \frac{(1)^2 + (-1)^2}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$$

Variance for the sample (3, 5) with mean 4

$$= \frac{(3-4)^2 + (5-4)^2}{2} = \frac{(-1)^2 + (1)^2}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$$

Variance for the sample (4, 5) with mean 4.5

$$= \frac{(4-4.5)^2 + (5-4.5)^2}{2} = \frac{(-0.5)^2 + (0.5)^2}{2} = \frac{0.25+0.25}{2} = 0.25$$

Variance for the sample (5, 5) with mean 5

$$=\frac{(5-5)^2+(5-5)^2}{2}=0$$

Variance for the sample (6, 5) with mean 5.5

$$=\frac{(6-5.5)^2+(5-5.5)^2}{2}=\frac{(0.5)^2+(-0.5)^2}{2}=\frac{0.25+0.25}{2}=0.25$$

Variance for the sample (3, 6) with mean 4.5

$$=\frac{(3-4.5)^2+(6-4.5)^2}{2}=\frac{(-1.5)^2+(1.5)^2}{2}=\frac{2.25+2.25}{2}=2.25$$

Variance for the sample (4, 6) with mean 5

$$=\frac{(4-5)^2+(6-5)^2}{2}=\frac{(-1)^2+(1)^2}{2}=\frac{1+1}{2}=1$$

Variance for the sample (5, 6) with mean 5.5

$$=\frac{(5-5.5)^2+(6-5.5)^2}{2}=\frac{(0.5)^2+(-0.5)^2}{2}=0.25$$

Variance for the sample (6, 6) with mean 6

$$=\frac{(6-6)^2+(6-6)^2}{2}=0$$

Thus the variance for each of the 16 samples

0	0.25	1	2.25
0.25	0	0.25	1
1	0.25	0	0.25
2.25	1	0.25	0

Thus the sampling distribution of variances (with replacement) is

S^2	0	0.25	1	2.25
Frequency	4	6	4	2

Mean of the sampling distribution of variance

$$=\frac{0\times4+0.25\times6+1\times4+2.25\times2}{16}$$

$$=\frac{10}{16}$$

$$=0.625$$

Variance of samplin

$$=\frac{4(0-0.62}{}$$

$$=\frac{4\times(0.625)}{}$$

$$=\frac{1.5625+0.}{}$$

$$=\frac{8.25}{16}$$

Variance = 0.5

Standard deviation o

$=\sqrt{0.5}$

Example 2.59. Finc variances for the po replacement.

Solution

The total number of s

$N_{C_n}=$

Here N = population

Total number of samp

(4, 5) (4, 6) (4,

4.5 5 5.

Variance of each sa

Variance for the samp

$$=\frac{(4-4.5)^2+}{2}$$

Variance for the samp

$$=\frac{(4-5)^2+(}{2}$$

Variance for the samp

$$=\frac{(4-5.5)^2+}{2}$$

Variance of sampling distribution of variance

$$= \frac{4(0-0.625)^2 + 6(0.25-0.625)^2 + 4(1-0.625)^2 + 2(2.25-0.625)^2}{16}$$

$$= \frac{4\times(0.625) + 6\times(0.140625) + 4\times(0.140625) + 2\times(2.640625)}{16}$$

0.25

$$= \frac{1.5625 + 0.84375 + 0.5625 + 5.2825}{16}$$

$$= \frac{8.25}{16}$$

2.25

Variance = 0.515625

Standard deviation of samples distribution of variance

$$= \sqrt{0.515625} = 0.71807$$

Ans.

Example 2.59. Find the mean and standard deviation of sampling distribution of variances for the population 4, 5, 6, 7 by drawing samples of size two without replacement.

Solution

The total number of samples without replacement is

$$N_{C_n} = 4_{C_2} = \frac{\angle 4}{\angle 2\ \angle 4-2} = \frac{\angle 4}{\angle 2\ \angle 2} = \frac{4\times 3\times \angle 2}{2\times 1\times \angle 2} = 6$$

Here N = population size and n = sample size

Total number of samples = 6

(4, 5)	(4, 6)	(4, 7)	(5, 6)	(5, 7)	(6, 7)
4.5	5	5.5	5.5	6	6.5

Variance of each sample

Variance for the sample (4, 5) with mean 4.5

$$= \frac{(4-4.5)^2 + (5-4.5)^2}{2} = \frac{(0.5)^2 + (0.5)^2}{2} = \frac{0.25+0.25}{2} = 0.25$$

Variance for the sample (4, 6) with mean 5

$$= \frac{(4-5)^2 + (6-5)^2}{2} = \frac{(-1)^2 + (1)^2}{2} = \frac{1+1}{2} = 1$$

Variance for the sample (4, 7) with mean 5.5

$$= \frac{(4-5.5)^2 + (7-5.5)^2}{2} = \frac{(1.5)^2 + (1.5)^2}{2} = \frac{2.25+2.25}{2} = 2.25$$

Variance for the sample (5, 6) with mean 5.5

$$=\frac{(5-5.5)^2+(6-5.5)^2}{2}=\frac{(0.5)^2+(0.5)^2}{2}=\frac{0.25+0.25}{2}=0.25$$

Variance for the sample (5, 7) with mean 6

$$=\frac{(5-6)^2+(7-6)^2}{2}=\frac{(-1)^2+(+1)^2}{2}=\frac{1+1}{2}=1$$

Variance for the sample (6, 7) with mean 6.5

$$=\frac{(6-6.5)^2+(7-6.5)^2}{2}=\frac{(0.5)^2+(0.5)^2}{2}=\frac{0.25+0.25}{2}=0.25$$

Thus the variance for each of the 6 samples

0.25	1	2.25
0.25	1	0.25

Thus the sampling distribution of variances (without replacement) is

S^2	0.25	1	2.25
Frequency	3	2	1

Mean of the sampling distribution of variance

$$=\frac{0.25\times 3+1\times 2+2.25\times 1}{6}$$

$$=\frac{0.75+2+2.25}{6}\qquad =\frac{5}{6}=0.8333$$

Variance of the sampling distribution of variance

$$=\frac{3\times(0.25-0.8333)^2+2(1-0.8333)^2+1(2.25-0.8333)^2}{6}$$

$$=\frac{3\times 0.34023+2\times 0.1667+1(2.007038)}{6}$$

$$=\frac{1.02071+0.05557+2.007038}{6}$$

$$=\frac{3.083318}{6}$$

Variance $= 0.513888$

Standard deviation of S.D of variance $=\sqrt{0.513888}$

$= 0.71685$ **Ans.**

Binomial Distribu

1) The probabili
10 cars are ex
(a) None wi
(b) Exactly
(c) At least t

[Hint: (a) P(x=0
(b) P(x
=
=
(c) P(x
=
=

2) Soft wares are
software does n
computer, find t
(a) Exactly tw
(b) At least tw
(c) All will wo

[Hint:(a) P(x=2)=b(2;10;

(b) P(x≥2)= 1- P(x
= 1- Σ
= 1- 0.1
(c)P(All will work

Poisson distribution

3) 2% of total pri
books are kept i
1) Two books w
2) At least three
3) 2<books has

= 0.25

= 0.25

s

$)^2$

EXERCISE

Binomial Distribution

1) The probability that a car manufactured by a company is defective is 0.5. if 10 cars are examined, find the probability that
   (a) None will be defective
   (b) Exactly five will be defective
   (c) At least three will be defective

[**Hint:** (a) $P(x=0)=b(0;10;0.5) = 0.0010$ **Ans.**

(b) $P(x=5)=b(5;10;0.5)$

$$= \sum_{x=0}^{5} b(x;10,0.5) - \sum_{x=0}^{4} b(x;10;0.5)$$

$= 0.6230-0.3770 = 0.2460$ **Ans.**

(c) $P(x\geq3) = 1-P(x<3)$

$$= 1- \sum_{x=0}^{4} b(x;\ 10;\ 0.5)$$

$= 1-0.0547 = 0.9453$ **Ans.**]

2) Soft wares are developed by a software company, the probability that software does not work properly is 0.4. if 10 such soft wares are Installed in a computer, find the probability that
   (a) Exactly two does not work properly
   (b) At least two does not work properly
   (c) All will work properly

[**Hint**:(a) $P(x=2)=b(2;10;0.3)= \sum_{x=0}^{2} b(x;10;0.3) - \sum_{x=0}^{1} b(x;10;0.3)$

$= 0.3823-0.1493 = 0.2335$ **Ans.**

(b) $P(x\geq2)= 1- P(x<2)$

$$= 1- \sum_{x=0}^{1} b(x;10;0.3)$$

$= 1- 0.1493 = 0.8507$ **Ans.**

(c)P(All will work properly)= P(zero software does not work properly) =P(x=0)

$=b(0;10;0.3) =0.0282$ **Ans.**]

Poisson distribution

3) 2% of total printed books in a printing press have printing mistakes, 100 books are kept in a box. What is the probability that there will be
   1) Two books which has printing mistakes
   2) At least three books which has printing mistakes
   3) 2<books has printing mistakes<5

[**Hint**: $\mu = np = 0.02 \times 100 = 2$

(1) P (2, 2) = 0.272 **Ans.**

(2) P ($x \geq 3$) = 1-P(x=0)-P(x=1)-P(x=2)

= 0.320 **Ans.**

(3)P (2<x<5) = 0.272 **Ans.**]

4) If x is a poisons variate such that P(x=1) =14 P(x=3) find the probability P(x=0)

[**Hint**: P(x=1)=14 P(x=3)

$\mu = \sqrt{3/7} = 0.654$

$P(x=0) = (e^{-\mu}\ \mu^{x})/x! = e^{-\mu} = 0.5199$ **Ans.**]

5) If x is a poisons variate such that P(x=1) = 7 P(x=3) find μ.

[**Hint**: P(x=1)=7 P(x=3)

$\mu = \sqrt{6/7} = 0.9258$ **Ans.**]

6) If two cards are drawn from a pack of 52 cards, which are hearts. Using Poisson distribution, determine the probability of getting two hearts at least three times in 51 consecutive trials of two cards drawing each time.

[**Hint**: $\mu = np = 51 \times (13c_2/52c_2) = 3$

P ($x \geq 3$) = 1-P(x=0) - P(x=1) - P(x=2) ≑ 0.5767 **Ans.**

Normal Distribution:

7) If x is a normal variate with mean 30 and standard deviation 5. Find the probability that

1) $26 \leq x \leq 30$ 2) $X \geq 45$

[**Hint**: 1) $Z = (x-\mu)/\sigma$, $Z_1 = -0.8$, $Z_2 = 0$

P ($-0.8 \leq z \leq 0$) =0.2881 **Ans.**

2) P($x \geq 45$)=P($z \geq 3$)=0.0013 **Ans.**]

8) The marks of students of an engineering college were found to be normally distributed with mean 50 marks and standard deviation 5 marks. Find the number of students who got the marks more than 60. If no of students are 700.

[**Hint**: P(x>60) = P(z>2) = 0.5-Area from 0 to 2

= 0.5-0.4772= 0.0228

The no of students who got more than 60% of marks

= 15.96 × 700

= 16 students **Ans.**]

Sampling Distribu

9) A populatior samples of : population. F

a) The mean

b) The standa

[**Hint**: a) Mean (

b) The s

$\sigma = \sqrt{\frac{(5-12.5)^2 + (}{}}$

10) A population samples of s population. Fi

a) The mean (

b) The standa

c) The mean (

d) The standa

[**Hint**: a) Mean o

b) The sta

$\sigma = \sqrt{\frac{(5-12.8}{}}$

$= \sqrt{42.96} =$

c) The total numbe

Here N = Pop

Listing all pos

Sampling Distribution

9) A population consists of four numbers 5, 10, 15, 20 consider all possible samples of size two which can be drawn without replacement from the population. Find

a) The mean of the population

b) The standard deviation of the population

[**Hint**: a) Mean of population $\mu = \dfrac{\sum x_i}{N} = \dfrac{5+10+15+20}{4} = \dfrac{50}{4} = 12.5$ **Ans.**

b) The standard deviation of the population $\sigma = \sqrt{\dfrac{\sum_{i=1}^{N}(x_i-\mu)^2}{N}}$

$$\sigma = \sqrt{\frac{(5-12.5)^2+(10-12.5)^2+(15-12.5)^2+(20-12.5)^2}{4}} = \sqrt{31.25} = 5.590 \text{ Ans.]}$$

10) A population consists of five numbers 5, 8, 12, 15,24 consider all possible samples of size two which can be drawn without replacement from the population. Find

a) The mean of the population

b) The standard deviation of the population

c) The mean of the sampling distribution of means

d) The standard deviation of sampling distribution of means.

[**Hint**: a) Mean of population $\mu = \dfrac{\sum x_i}{N} = \dfrac{5+8+12+15+24}{5} = \dfrac{64}{5} = 12.8$ **Ans.**

b) The standard deviation of the population $\sigma = \sqrt{\dfrac{\sum_{i=1}^{N}(x_i-\mu)^2}{N}}$

$$\sigma = \sqrt{\frac{(5-12.8)^2+(8-12.8)^2+(12-12.8)^2+(15-12.8)^2+(24-12.8)^2}{5}}$$

$= \sqrt{42.96} = 6.5543$ **Ans.**

c) The total number of samples without replacement is , $N_{c_n} = 5_{c_2} = 10$,

Here N = Population size, n = Sample size

Listing all possible samples of size 2 from given population without replacement me get

(5, 8) (5, 12) (5, 15) (5, 24)(8, 12)(8, 15) (8, 24)(12, 15)(12, 24)(15, 24)

The mean for each of these samples,6.5,8.5,10,14.5,10,11.5,16,13.5,18,19.5

Sampling distribution of the mean

Sample mean	$\overline{X}_i$	6.5	8.5	10	11.5	13.5	14.5	16	18	19.5
Freq.	f_i	1	1	2	1	1	1	1	1	1

Mean of the sampling distribution of mean $\mu_{\overline{x}} = \dfrac{128}{10} = 12.8$ **Ans.**

d)The standard deviation of sampling distribution of mean

$$\sigma_{\overline{x}}^2 = \frac{\sum_{i=1}^{N} f_i (X_i - \mu)^2}{N} = \frac{202.2}{10} = 16.11$$

$\sigma_{\overline{x}} = 4.0136$ **Ans.]**

11) Let $U_1 = (1,7,8), U_2 = (3,4)$ find

a. μ_{U_1} b. μ_{U_2} c. $\mu_{U_1 - U_2}$

d. σ_{U_1} e. σ_{U_2} f. $\sigma_{U_1 - U_2}$

[Hint: U_1-U_2={-2,-3,4,3,5,4}

a $\mu_{U_1} = \dfrac{16}{3} = 5.33$ **Ans.**, b. $\mu_{U_2} = \dfrac{7}{2} = 3.5$ **Ans.**

c. $\mu_{U_1 - U_2} = \mu_{U_1} - \mu_{U_2} = \dfrac{16}{3} - \dfrac{7}{2} = \dfrac{11}{6}$ **Ans.**

d. $\sigma_{U_1}^2$ = Variance of the population $U_1 = \dfrac{(1-5.33)^2 + (7-5.33)^2 + (8-5.33)^2}{3}$

$\sigma_{U_1}^2 = 9.5555$, $\sigma_{U_1} = 3.0912$ **Ans.**

e. $\sigma_{U_2}{}^2$ = Variance of the population $U_2 = \dfrac{(3-3.5)^2 + (4-3.5)^2}{2}$

$\sigma_{U_2}{}^2 = 0.25$, $\sigma_{U_2} = 0.5$ **Ans.**

f $\sigma_{U_1 - U_2} = \sqrt{\sigma_{U_1}^2 + \sigma_{U_2}^2} = \sqrt{9.5555 + 0.25}$ $= 3.1313$ **Ans.]**

OBJECTIVE TYI

1. The mean of tl
(a) np

Ans. (a)

2. If n and p a
deviation of th
(a) np

Ans. (b)

3. The distributio
(a) Normal dist
(c) Poisson dist

Ans. (a)

4. The mean and
(a) Equal
(c) Not Equal

Ans. (a)

5. The mean of a l
(a) e^{-2}

Ans. (b)

6. The variance of
(a) 3

Ans. (a)

7. If P (1) =P (2), t
(a) 3

Ans. (c)

8. If the variance o
(a) e^{-2}

Ans. (a)

OBJECTIVE TYPE QUESTIONS

1. The mean of the binomial distribution is

   (a) np (b) p/q (c) npq (d) none of these

Ans. (a)

2. If n and p are the parameters of a binomial distribution, the standard deviation of this distribution

   (a) np (b) $\sqrt{npq}$ (c) npq (d) none of these

Ans. (b)

3. The distribution in which mean, median and mode are equal is

   (a) Normal distribution (b) Binomial distribution
   (c) Poisson distribution (d) none of these

Ans. (a)

4. The mean and variance of a Poisson distribution is

   (a) Equal (b) Some times may be equal
   (c) Not Equal (d) none of these

Ans. (a)

5. The mean of a Poisson variate X is one then $P(X=1)$ is

   (a) e^{-2} (b) e^{-1} (c) e^{-3} (d) none of these

Ans. (b)

6. The variance of a Poisson distribution with parameter $\lambda=3$ is

   (a) 3 (b) 4 (c) 2 (d) none of these

Ans. (a)

7. If $P(1) = P(2)$, then the mean of a Poisson distribution is

   (a) 3 (b) 4 (c) 2 (d) none of these

Ans. (c)

8. If the variance of a Poisson variate is 2 then $P(X=0)$ is

   (a) e^{-2} (b) e^{-1} (c) e^{-3} (d) none of these

Ans. (a)

9. The area under the whole normal curve is
   (a) Unity (b) more than unity (c) Less than 1
   (d) none of these

Ans. (a)

10. In the standard normal curve the area between z= -1 and z= 1 is nearly
    (a) 68% (b) 80% (c) 70% (d) none of these

Ans. (a)

11. The Probability of getting one girl in a family of 5 children is
    (a) ½ (b) 2/3 (c) 1/5 (d) 1/4

Ans. (c)

12. Mean of the Binomial Distribution is 6 and variance is 3, then n =
    (a) 1 (b) 3 (c) 9 (d) 12

Ans. (d)

13. The Probability of getting 3 heads in tossing 5 coins is
    (a) 1/16 (b) 3/16 (c) 5/16 (d) 3/32

Ans. (c)

14. A coin is tossed n times the probability at x = 2 = 3/8
    (a) 1 (b) 2 (c) 3 (d) 4

Ans. (d)

15. A coin is tossed 5 times then p(x = 2) =
    (a) 1/8 (b) 5/16 (c) 5/8 (d) 7/8

Ans. (b)

16. Collection of all objects, under study is known as.
    (a) Population (b) Sample (c) Parameter (d) none of these

Ans. (a)

17. The statistical
    (a) Sample

Ans. (c)

18. The probabilit
    (a) Normal dis
    (c) Poisson di

Ans. (d)

19. A finite subset
    (a) Sample

Ans. (a)

20. The total numl
    be drawn from
    (a) N_{C_n}

Ans. (a)

21. The total numl
    be drawn from
    (a) N_{C_n}

Ans. (b)

22. The finite popu
    (a) $\frac{n-N}{N-1}$

Ans. (b)

23. If Sampling is
    (a) $\mu_{\bar{X}} = \mu,$

    (c) $\mu_{\bar{X}} = \mu,$

Ans. (b)

17. The statistical constants of the population are called

(a) Sample (b) Population Statistic (c) Parameter (d) none of these

Ans. (c)

18. The probability distribution of mean is called

(a) Normal distribution (b) Binomial distribution

(c) Poisson distribution (d) Sampling distribution

Ans. (d)

19. A finite subset of the population known as sample.

(a) Sample (b) Population Statistic (c) Parameter (d) none of these

Ans. (a)

20. The total number of all possible samples each of the same size n, which can be drawn from the population of size N without replacement, is given by

(a) N_{C_n} (b) N (c) $1/N_{C_n}$ (d) none of these

Ans. (a)

21. The total number of all possible samples each of the same size n, which can be drawn from the population of size N with replacement, is given by

(a) N_{C_n} (b) N^n (c) $1/N_{C_n}$ (d) none of these

Ans. (b)

22. The finite population correction factor is known as.

(a) $\dfrac{n-N}{N-1}$ (b) $\dfrac{N-n}{N-1}$ (c) $\dfrac{n-1}{N-1}$ (d) none of these

Ans. (b)

23. If Sampling is done with replacement then

(a) $\mu_{\overline{X}} = \mu, \quad \sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{N}}$ (b) $\mu_{\overline{X}} = \mu, \quad \sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$

(c) $\mu_{\overline{X}} = \mu, \quad \sigma_{\overline{X}} = \dfrac{\sigma^2}{\sqrt{n}}$ (d) none of these

Ans. (b)

24. If Sampling is done without replacement then

(a) $\mu_{\overline{X}} = \mu, \sigma_{\overline{X}} = \dfrac{\sigma}{N}\sqrt{\dfrac{N-n}{N-1}}$ (b) $\mu_{\overline{X}} = \mu, \sigma_{\overline{X}} = \dfrac{\sigma^2}{\sqrt{N}}\sqrt{\dfrac{N-n}{N-1}}$

(c) $\mu_{\overline{X}} = \mu, \sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{N}}\sqrt{\dfrac{N-n}{N-1}}$ (d) none of these

Ans. (c)

26. Sample of size 2 taken from the population 62,57,63,58 Variance of the mean of sampling distribution is

(a) 1.17 (b) 2.17 (c) 3.17 (d) 4.17

Ans. (b)

27. If sample of size 2 are taken with replacement from the population 2, 4, 6, 8, 10 then the mean of the mean of sample distribution is

(a) 1 (b) 3 (c) 5 (d) 6

Ans. (d)

28. In above question, the sample of size 2 are taken without replacement, then mean of means of sampling distribution is

(a) 1 (b) 3 (c) 5 (d) 6

Ans. (d)

29. If a sample of size 64 is taken from a population whose standard deviation is 4, then standard error & Probable error is

(a) 0.5, 0.352 (b) 0.5, 0.337 (c) 0.5, 0.321 (d) 0.49, 0.337

Ans. (b)

30. If the size of the sample is 5 and size of the population is 2000. The correction factor is

(a) 0.555 (b) 0.888 (c) 0.444 (d) 0.127

T

"Th

3.1 ESTIMATE

To find an un
estimate.

3.2 ESTIMATOR

The method o
estimator.

For example s
mean is a method t

A population
be selected such th
done in two ways.

1. Point estimati
2. Interval estim

An estimate o
point estimate of th

UNIT-3

TESTING OF HYPOTHESIS-I

"Things don't change you change your way of looking, that's all."

3.1 ESTIMATE

To find an unknown population parameter, a statement is made which is an estimate. **(Nov.2009 set 1)**

3.2 ESTIMATOR

The method or rule to determine an unknown population parameter is called an estimator.

(Nov.2009 set 1)

For example sample mean is an estimator of population mean because sample mean is a method to determine of determining the population mean.

A population can have many or one, or two estimators. The estimators should be selected such that they are nearer to the parameter value. The estimation can be done in two ways.

1. Point estimation **(Nov.2009 set 1, set 3)**
2. Interval estimation **(Nov.2009 set 3)**

An estimate of a population parameter given by a single number is called a point estimate of the parameter. An estimate of a population parameter given by two

numbers between which the parameter may be considered to lie is called an interval estimate of the parameter.

Example : If we say that the cost of bike is 25,000, we are giving a point estimate. If, on the other hand, we say that the cost of bike is 25,000 to 35,000 i.e. the cost lies between 25,000 to 35,000, we are giving an interval estimate.

3.3 UNBIASED ESTIMATOR

A statistic $\hat{\theta}$ is known as an unbiased estimator of the corresponding parameter θ if

$$E(\hat{\theta}) = E(\text{statistic}) = \text{parameter} = \theta$$

i.e. the mean of the sampling distribution of estimator equals to θ.

3.4 MAXIMUM ERROR OF ESTIMATE E

Since the sample mean estimate is not always equals to the mean of population μ. The error, $\bar{x}$ -μ, is the difference between the estimator and the quantity which is supposed to estimate.

To examine, this error let us make use of the fact that for large n

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$$

Is a random variable having approximately the standard normal distribution?

We can assert with probability (1 – α) that the inequality

$$-z_{\alpha/2} \leq \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}$$

Will be satisfied or

$$|\bar{x}-\mu| \leq z_{\alpha/2}.\sigma/\sqrt{n}$$

E stand for the maximum of these values of $|\bar{x}-\mu|$, the maximum error of estimate

$$E = z_{\alpha/2}.\sigma/\sqrt{n}$$

3.5 CONFIDENCE INTERVAL FOR μ

A (1 – α)100% confidence internal for μ is given by

$$\bar{x} - z_{\alpha/2}\ \sigma/\sqrt{n} < \mu < \bar{x} + z_{\alpha/2}\ \sigma/\sqrt{n}$$

Thus with the po
One can assert w
$z_{\alpha/2}\ \sigma/\sqrt{n}$ for
The maximum err

With (1 – α) prob

3.6 SAMPLE SI
The formula

Solving this equati

3.7 INTERVAL
Point estimat
intended to estima
Which determines
sample of size (n
variance σ^2. Then t

Confidence interva

When (n < 30), ass

3.8 STATISTICA
We know th
estimation but to d
we make certain as
be true or may not
To estimate
parameter. The hyp

Thus with the population mean μ, sample mean $\bar{x}$ for a large sample $(n \geq 30)$. One can assert with probability (1-α) that the errors $|\bar{x} - \mu|$ will be less than $z_{\alpha/2}\ \sigma/\sqrt{n}$ for the small samples if σ is unknown or (n < 30), σ is replaced by S.

The maximum error estimate

$$E = t_{\alpha/2}\ S/\sqrt{n}$$

With (1 – α) probability, t-distribution is with (n-1) degrees of freedom.

3.6 SAMPLE SIZE

The formula for maximum error can also be used to determine the sample size

$$E = z_{\alpha/2}\ \sigma/\sqrt{n}$$

Solving this equation for n, we get

$$n = \left[\frac{z_{\alpha/2}.\sigma}{E}\right]^2$$

3.7 INTERVAL ESTIMATION

Point estimates can not be expected to coincide with the quantities they are intended to estimate, it is sometimes preferable to replace with interval estimates. Which determines an interval in which the parameter lies. Consider a large random sample of size (n ≥ 30) from a population with unknown mean μ and known variance σ^2. Then the large sample confidence interval for μ.

$$\bar{x} - z_{\alpha/2}\,\sigma/\sqrt{n} < \mu < \bar{x} + z_{\alpha/2}\ \sigma/\sqrt{n}$$

Confidence interval for μ for small samples

When (n < 30), assuming sampling from normal population

$$\bar{x} - t_{\alpha/2}\, s/\sqrt{n} < \mu < \bar{x} + t_{\alpha/2}\ s/\sqrt{n}$$

3.8 STATISTICAL HYPOTHESIS

We know that to find an unknown population parameter, the method is called estimation but to decide about population parameter on the basis of simple statistic, we make certain assumption about the populations. These assumptions which may be true or may not be true are called statistical hypothesis.

To estimate the value of a parameter we must test a hypothesis about a parameter. The hypothesis being tested is referred as H.

3.9 NULL HYPOTHESIS

The hypothesis formulated for the rejecting it or hypothesis which is tested for possible rejection under the assumption that it is true is called the null hypothesis and is denoted by H_0.

3.10 ALTERNATIVE HYPOTHESIS

Any hypothesis which is complementary to the null hypothesis is called an alternative hypothesis and is denoted by H_1 when we test a hypothesis, there are two possibilities, H is true or false. If Hypothesis H is true and accepted or false and rejected in both cases decision is correct.

3.11 TYPE – I ERROR **(Nov.2009 set 3)**
(R05, Supple.Feb.2010 Set.3)

If the hypothesis H is true but rejected, it is rejected in error and it is called type – I error or we can say rejection of the null hypothesis when it is true is called a type I error.

3.12 TYPE – II ERROR **(Nov.2009 set 3)**
(R05, Supple.Feb.2010 Set.3)

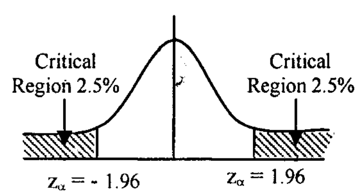
If the hypothesis H is false but accepted, it is accepted in error and it is called type – II error or we can say acceptance of the null hypothesis when it is false is called a type II error.

3.13 LEVEL OF SIGNIFICANCE **(Nov.2009 set 1, set 3)**

The probability level below which we reject the hypothesis is known as the level of significance or the probability of the value of the variate falling in the critical region is the level of significance.

3.14 CRITICAL REGION

The critical region is that in which a sample value falling is rejected. Usually we take two critical regions which cover 5% and 1% areas of the normal curve.



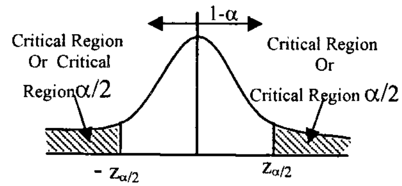
3.15 TWO TAILED TEST

If the alternative hypothesis H_1 is the type of not equal to then the critical region lies on the both sides, right and left, as shown in figure.

3.16 RIGHT ONE
If the alterna
region lies on the ri

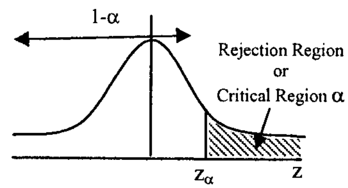
3.17 LEFT ONE T
If the alternati
lies on the left side,

3.18 TEST OF TH
Working Procedur
1. Null Hypothesis
2. Alternative Hyp
3. Level of signific
4. Critical regio
Hypothesis wh
if it is greater t
then test will b
test. Critical re
5. Test of Statisti
statistics can b



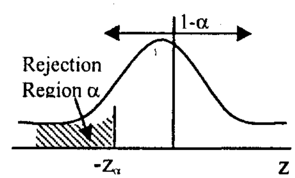
3.16 RIGHT ONE TAILED TEST

If the alternative hypothesis H_1 is the type of greater than, then the critical region lies on the right side, as shown in figure.



3.17 LEFT ONE TAILED TEST

If the alternative hypothesis H_1 is the type of less than, then the critical region lies on the left side, as shown in figure.



3.18 TEST OF THE HYPOTHESIS (Nov.2009 set 4)

Working Procedure

1. Null Hypothesis H_0
2. Alternative Hypothesis H_1
3. Level of significance : α
4. Critical region: Critical region is decided, according to the alternative Hypothesis which is the greater than type or less than type or not equal to type, if it is greater than type then test will be right one tail test. If it is less than type then test will be left one tail test. If it is not equal to type, test will be two tail test. Critical region will be taken according to, level of significance (α).
5. Test of Statistics: For the large samples and for the small samples, different statistics can be used.

6. Decision: Null Hypothesis is accepted or rejected it can be decided by relation between Z and Z_α (or $z_{\alpha/2}$) for large samples or t and t_α (or $t_{\alpha/2}$) for small samples.

3.19 TEST OF HYPOTHESIS (LARGE SAMPLES)

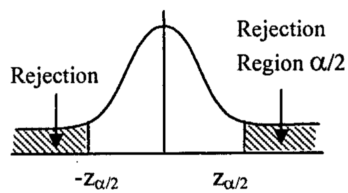
1. Single population mean μ, with known variance σ^2

   Working procedure

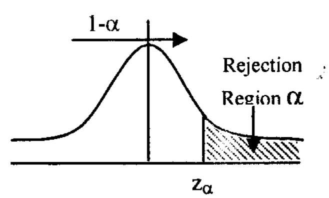
   For the large sample ($n \geq 30$), to test whether the population mean μ equals to a constant μ_0 or not, formulate

1. Null Hypothesis $H_0 : \mu = \mu_0$
2. Alternative Hypothesis $H_1 : \mu \neq \mu_0$ or $\mu > \mu_0$ or $\mu < \mu_0$
3. Level of significance : α
4. Critical region :

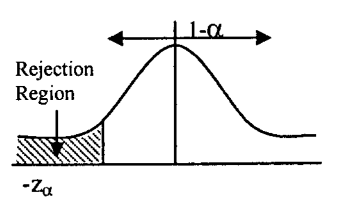
a. If $\mu \neq \mu_0$ test is two tail test for given α, critical values are $-z_{\alpha/2}$ and $+z_{\alpha/2}$ from the normal distribution table.



b. If $\mu > \mu_0$ test is right one tail test for given α, critical values is z_α, from the normal distribution table..



c. If $\mu < \mu_0$ test is left one tail test for given α, critical value is $-z_\alpha$, from the normal distribution table.



5. Test of statis

$Z =$

Where $\overline{x}$ is

μ is p

σ is p

n sam

For the larg deviation.

6. Decision

a. If $-z$

Hypot

b. $z_\alpha >$

c. $z > -$

3.20 TEST OF H

Concerning two m

Working Procedu

For the large samp whether

1. Null Hypothesi
2. Alternative Hy
3. Level of signifi
4. Critical region

a. If μ_1-$\mu_2 \neq 0$ t

$z_{\alpha/2}$ from the

5. Test of statistic

$$Z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$$

Where $\overline{x}$ is mean of the sample

μ is population mean

σ is population standard deviation

n sample size

For the large sample if σ is unknown, it is replaced by sample standard deviation.

6. Decision

a. If $-z_{\alpha/2} < z < z_{\alpha/2}$

Hypothesis is accepted for T.T.T. otherwise rejected.

b. $z_{\alpha} > z$ Hypothesis is accepted otherwise rejected for R.O.T.T.

c. $z > -z_{\alpha}$ Hypothesis is accepted otherwise rejected for L.O.T.T.

3.20 TEST OF HYPOTHESIS (LARGE SAMPLES)

Concerning two means, with known variances σ_1 and σ_2 **(Nov.2009 set 1)**

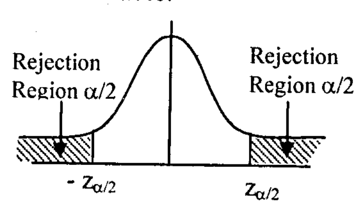
Working Procedure

For the large samples (n_1, $n_2 \geq 30$) to test the hypothesis for difference of means, whether

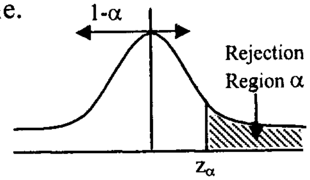
$$\mu_1 - \mu_2 = \delta = \text{constant} = 0 \text{ or not. Then formulate}$$

1. Null Hypothesis $H_0 : \mu_1 - \mu_2 = \delta = 0$
2. Alternative Hypothesis $H_1 : \mu_1 - \mu_2 \neq 0$ or $\mu_1 - \mu_2 > 0$ or $\mu_1 - \mu_2 < 0$
3. Level of significance : α
4. Critical region :

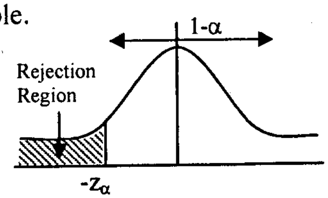
a. If $\mu_1 - \mu_2 \neq 0$ test is two tail test. For given α, critical values are $-z_{\alpha/2}$ and $z_{\alpha/2}$ from the normal distribution table.



b. If $\mu_1 - \mu_2 > 0$ test is right one tail test, for given α critical value is z_α from the normal distribution table.



c. $\mu_1 - \mu_2 < 0$ test is left one tail test, for given α. Critical value is $-z_\alpha$ from the normal distribution table.



5. Test of statistic

The statistic for test concerning difference between two means

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$\sigma_1^2 \rightarrow$ First population variance

$\sigma_2^2 \rightarrow$ Other population variance

$\bar{x}_1 \rightarrow$ First sample mean

$\bar{x}_2 \rightarrow$ other sample mean

$n_1 \rightarrow$ first sample size

$n_2 \rightarrow$ second sample size

For the large samples $(n_1, n_2 \geq 30)$ σ_1^2 *and* σ_2^2 are unknown, it can be replaced by sample variances S_1^2 and S_2^2 then the statistic is

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

6. Decision

a. If $-z_{\alpha/2} < z < z_{\alpha/2}$ Hypothesis is accepted for two tail test otherwise rejected.

b. $z_\alpha > z$ Hypothesis is accepted for right one tail test otherwise rejected.

c. $z > -z_\alpha$ Hypothesis is accepted for left one tail test otherwise rejected.

Example 3.1. It is
continuous use until
assumed that σ= 48 h
to assert with 90% co

Solution: Given t
$\sigma = 48$ h
$z_{\alpha/2} = 1$
$\alpha = 90\%$
Maximu
Sample s
$n =$
$=$
$=$
$= ($

Example 3.2. In a
of 80 body repair cos
62.35. If $\bar{x}$ is used a
confidence we can ass
the confidence internal

Solution: Given that
Mean of t
Standard d
Maximum
Maximum

SOLVED EXAMPLES

Example 3.1. It is desired to estimate the mean number of hours of continuous use until a certain computer will first require repairs. If it can be assumed that σ= 48 hours, how large a sample be needed so that one will be able to assert with 90% confidence that the sample mean is off by at most 10 hours.

(Supple. Nov. / Dec. 2004 Set 3)

Solution: Given that

$\sigma = 48$ hours

$z_{\alpha/2} = 1.645$ For confidence level

$\alpha = 90\%$

Maximum error E = 10 hours

Sample size n =?

$$n = \left[\frac{z_{\alpha/2}\,\sigma}{E}\right]^2$$

$$= \left[\frac{1.645 \times 48}{10}\right]^2$$

$$= \left[\frac{78.96}{10}\right]^2$$

$$= (7.896)^2 = 62.3464 = 63$$

Ans.

Example 3.2. In a study of an automobile insurance a random sample of 80 body repair costs had a mean of Rs. 472.35 and a standard deviation of Rs. 62.35. If $\overline{x}$ is used as a point estimate to the true average repair costs, with what confidence we can assert that the maximum error does not exceed Rs.10? Find also the confidence internal with that confidence.

(Supple. Feb.2007 Set 4)

Solution: Given that, Sample size = 80

Mean of the sample $\overline{x} = 472.35$

Standard deviation of the sample S = 62.35

Maximum error E = 10

Maximum error $E = z_{\alpha/2} \,.\, \sigma / \sqrt{n}$

$$10 = z_{\alpha/2} \; \frac{62.35}{\sqrt{80}}$$

$$\frac{10 \times 8.944}{62.35} = z_{\alpha/2}$$

$$1.43 = z_{\alpha/2}$$

With 85% confidence we can assert that the maximum error doesn't exceed is 10.

$\bar{x} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$ and $\bar{x} + z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$ is the $(1-\alpha)$ 10 %. Confidence interval for the population mean.

$$\bar{x} - z_{\alpha/2}\left(\sigma/\sqrt{n}\right) < \mu < \bar{x} + z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$$

$$472.35 - \frac{(1.44)(62.35)}{\sqrt{80}} < \mu < 472.35 + \frac{(1.44)(62.35)}{\sqrt{80}}$$

$$472.35 - \frac{89.784}{\sqrt{80}} < \mu < 472.35 + \frac{89.784}{\sqrt{80}}$$

$$472.35 - \frac{89.784}{8.9442} < \mu < 472.35 + \frac{89.784}{8.9442}$$

$$472.35 - 10.0382 < \mu < 472.35 + 10.0382$$

$$462.311 < \mu < 482.388$$

Confidence interval is (462.311, 482.388) **Ans.**

Example 3.3. A research worker wishes to estimate mean of a population by using sufficiently large sample. The probability is 95% that mean will not differ from the true mean by more than 25 percentage of the standard deviation. How large a sample should be taken? **(JNTU 1999)**

Solution: **Given that $z_{\alpha/2} = 1.96$ for confidence level $\alpha = 95\%$**

$$\left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| = z_{\alpha/2}$$

$$\left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| = 1.96$$

$$|\bar{x}-\mu| = 1.96\ \sigma/\sqrt{n} \quad \ldots\ldots\ldots (1)$$

Given that mean will not differ from the true mean by more than 25 percentage means

$$|\bar{x}-\mu| < \sigma/4 \ldots\ldots\ldots (2)$$

From equation (1) and (2)

$$1.96\frac{\sigma}{\sqrt{n}} < \frac{\sigma}{4}$$

$$1.96 < \frac{\sqrt{n}}{4}$$

$$1.96 \times 4 < \sqrt{n}$$

$$7.84 < \sqrt{n}$$

$$61.46 < n$$

$n > 62$ approximately **Ans.**

Example 3.4. A random sample of size 100 has a standard deviation of 5. What can you say about the maximum error with 95% confidence?

(Supple. Feb. 2007 Set 3)
(Supple. Nov. 2006 Set 1)

Solution: Given that

$n = 100$

Standard deviation $\sigma = 5$

Maximum error =?

$z_{\alpha/2} = 1.96$ For 95% confidence interval

$$\text{Maximum error} = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$= 1.96 \times \frac{5}{\sqrt{100}}$$

$$= 1.96 \times \frac{5}{10}$$

$= 0.98$ **Ans.**

Example 3.5. Find the size of the sample if the S.D. of the population is 9 and there should be 99% confidence that the error of estimate will not exceed 3.

(Nov. 2006)

Solution: Given that

$\sigma = 9$

$n = ?$

M.E. = 3

$z_{\alpha/2} = 2.58$ for confidence level $\alpha = 99\%$

$$n = \left(\frac{z_{\alpha/2}\,\sigma}{E}\right)^2$$

$$= \left(\frac{2.58 \times 9}{3}\right)^2$$

$$= (7.74)^2$$

$$n = 59.907$$ **Ans.**

Example 3.6. A random sample of size 81 was taken whose variance is 20.25 and mean 32 construct 98% confidence interval. **(Supple. Nov. /Dec. 2005 Set 2)**

Solution: Given that

Sample size n = 81

Variance $\sigma^2 = 20.25$, $\sigma = 4.5$

Mean $\overline{X} = 32$

$z_{\alpha/2} = 2.33$ for 98% level of significance

Confidence interval =?

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$32 - \frac{2.33 \times 4.5}{\sqrt{81}} < \mu < 32 + \frac{2.33 \times 4.5}{\sqrt{81}}$$

$$32 - \frac{10.485}{9} < \mu < 32 + \frac{10.485}{9}$$

$$32 - 1.165 < \mu < 32 + 1.165$$

$$30.835 < \mu < 33.165$$

Confidence interval (30.835, 33.165) **Ans.**

Example 3.7. The mean and the standard deviation of a population are11,795 and 14,054 respectively. If n = 50 final 95% confidence interval for the mean.

(Reg. April/May 2005 Set 4)

Solution: Given that

Mean of the population $\mu = 11{,}795$

Standard deviation of the population $\sigma = 14{,}054$ n = 50

$z_{\alpha/2} = 1.96$ for $\alpha = 95\%$ confidence level

Confidence interval =?

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$11795 - \frac{1.96 \times 14054}{\sqrt{50}} < \mu < 11795 + \frac{1.96 \times 14054}{\sqrt{50}}$$

$$11795 - \frac{27545.84}{7.0710} < \mu < 11795 + \frac{27545.84}{7.0710}$$

$$11795 - 3895.60 < \mu < 11795 + 3895.60$$

$$7899.39 < \mu < 15690.60$$

Confidence level (7899.39, 15690.60) **Ans.**

Example 3.8. What is the maximum error one can expect to make with probability 0.90 when using the mean of a random sample of size n = 64 to estimate the mean of a population with $\sigma^2 = 2.56$?

(Supple. Nov. /Dec. 2004 Set 4)

Solution: Maximum error $E = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

$z_{\alpha/2} = 1.645$ for 90% confidence

$$\sigma^2 = 2.56$$

$$\sigma = \sqrt{2.56} = 1.6$$

$$E = 1.645 \times \frac{1.6}{\sqrt{64}} = \frac{2.632}{8}$$

$$E = 0.329$$

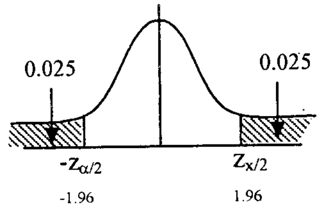
Ans.

Example 3.9 An oceanographer wants to check whether the average depth of the ocean in a certain region is 57. 4 fathoms, as had previously been recorded. What can be conclude at the level of significance α = 0.05 if soundings taken at 40 random locations in the given region yielded a mean of 59.1 fathoms with a standard deviation of 5.2 fathoms?

(Reg. April / May 2004 Set C)

Solution

1. Null hypothesis: $\mu = 57.4$
2. Alternative hypothesis: $\mu \neq 57.4$
3. Level of significance: $\alpha = 0.05$
4. Critical region: $Z_{\alpha/2} = 1.96$ for α = 0.05 level of significance. Test is T.T.T.



5. Test of statistic:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

Mean of sample $\bar{x}$ = 59.1
n = 40

standard deviation of sample : S = 5.2

$$z = \frac{59.1 - 57.4}{5.2/\sqrt{40}} = 2.067$$

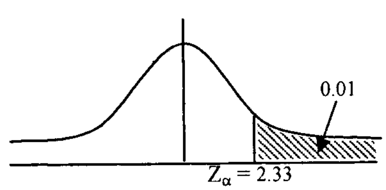
7. Decision : $-z_{\alpha/2} < z < z_{\alpha/2}$ then Null hypothesis is accepted
$-2.067 < 1.96 < +2.067$ Null hypothesis is accepted

Example 3.10. According to the norms established for a mechanical aptitude test. Persons who are 18 years old have an average height of 73.2 with a standard deviation of 8.6. If 45 normally selected persons of that age averaged 76.7. Test the null hypothesis μ = 73.2 against the alternative hypothesis μ > 73.2 at the 0.01 level of significance.

(Reg. April / May 2005)

Solution

1. Null hypothesis: μ = 73.2
2. Alternative hypothesis: μ > 73.2
3. Level of significance: α = 0.01
4. Critical region: $Z_{\alpha/2}$ =2.33 for α = 0.01 level of significance. Test is T.T.T.



5. Test of Statistic:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

Sample m
Populatioı
Standard (
Sample si:

6. Decision:
Null hypo

Example 3.11. A :
be regarded as a sa
standard deviation 25

Solution
1. Null hypo
2. Alternativ
3. Level of si
4. Critical re
$z_{\alpha/2}$ = 1.9
For α = 0.
5. Test of sta
Sample me
Population
Standard d

$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{}}$

$= \frac{70 - 6}{25/\sqrt{7}}$

= 1.673

6. Decision:
- 1.96

Example 3.12. A lad
118 words per minute
demonstrates a mean

Sample mean $\bar{x} = 76.7$
Population mean $\mu = 73.2$
Standard deviation $\sigma = 8.6$
Sample size $n = 45$

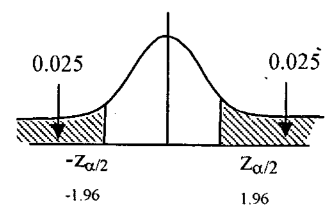
$$z = \frac{76.7 - 73.2}{8.6/\sqrt{45}} = \frac{3.5}{1.2820} = 2.7301$$

6. Decision: $z < z_\alpha$ then null hypothesis accepted but 2.7301>2.33
   Null hypothesis is rejected at Level of significance 0.01

Example 3.11. A sample of 64 students has a mean weight of 70 k.gms. Can this be regarded as a sample from a population with mean weight 65 k.gms. and standard deviation 25 k.gms.

(Reg. Nov. 2006 Set 3)

Solution



1. Null hypothesis H_0: $\mu = 65$ k.gms.
2. Alternative hypothesis H_1: $\mu \neq 65$
3. Level of significance: $\alpha = 0.05$
4. Critical region:
   $z_{\alpha/2} = 1.96$
   For $\alpha = 0.05$ level of significance
5. Test of statistic:
   Sample mean $\bar{x} = 70$ k.gms.
   Population mean $\mu = 65$ k.gms.
   Standard deviation $\sigma = 25$ k.gms.

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$= \frac{70 - 65}{25/\sqrt{70}} = \frac{5}{25/8.366} = \frac{5}{2.988}$$

$$= 1.673$$

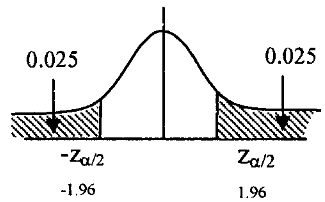
6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted
   $-1.96 < 1.673 < 1.96$ null hypothesis is accepted

Example 3.12. A lady stenographer claims that she can take dictation at the rate of 118 words per minute can we reject her claim on the basis of 100 trails in which she demonstrates a mean of 116 words and a S.D. of 15 words.

(Reg. Nov. 2006 Set 4, Set 3)

Solution

1. Null hypothesis H_0: lady stenographer can take the dictation at the rate of 118 words
2. Alternative hypothesis H_1: $\mu \neq 118$
3. Level of significance: $\alpha = 0.05$
4. Critical region: The test is two tailed test since alternative hypothesis is $\neq$ type, the critical region is, $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance.
5. Test of statistic: $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$



n = sample size = 100
σ = standard deviation = 15
μ = mean of the population = 118 words
$\bar{x}$ = mean of the sample = 116 words

$$z = \frac{116 - 118}{15/\sqrt{100}} = \frac{-2}{15/10} = \frac{-2}{1.5} = 1.333$$

$z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted

   $-1.96 < 1.333 < 1.96$ null hypothesis is accepted

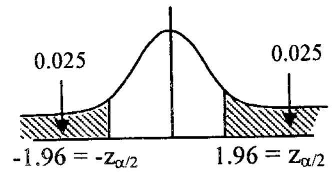
Example 3.13 Samples of students were drawn from two universities and from their weights in kilograms and standard deviation are calculated. Make a large sample test to test the significance of the difference between the means

(Supple. Nov. / Dec. 2005 Set 4)

	Mean	S.D	Size of the sample
University A	55	10	400
University B	57	15	100

Solution

1. Null hypothesis H_0: $\mu_1 = \mu_2$
2. Alternative hypothesis H_1: $\mu_1 \neq \mu_2$
3. Level of significance: $\alpha = .05$
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = .05$ level of significance



5. Test of statistic:

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

$\bar{x}_1$ = mean of the 1st sample = 55

$\bar{x}_2$ = mean of the 2nd sample = 57

S_1^2 = variance of the 1st sample = 100

S_2^2 = variance of the 2nd sample = 225

n_1 = size of the first sample = 400

n_2 = size of the seemed sample = 100

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$$

Note : When two variances are unknown $\left(\sigma_1^2, \sigma_2^2\right)$ they can be replaced by sample variances

$$Z = \frac{(55 - 57) - 0}{\sqrt{\dfrac{100}{400} + \dfrac{225}{100}}}$$

$$= \frac{-2}{\sqrt{\dfrac{1}{4} + \dfrac{9}{4}}} = \frac{-2 \times 2}{\sqrt{10}}$$

$$= \frac{-4}{3.162} = -1.265$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted

- 1.96 < -1.265 < 1.96 i.e. null hypothesis is accepted

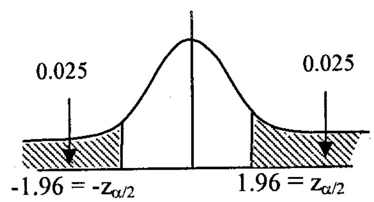
There is no significant difference between the means.

Example 3.14. A point manufacturer claims that the average drying time of his new "fast – drying" paint is 20 minutes and that a government agency wants to test the validity of this claim. Suppose, furthermore that 36 boards painted, respectively, with paint from 36 different one – gallon cans of this paint dried on the average in 20.75 minutes. Is this sufficient evidence to take appropriate action against the paint manufacturer? Justify. (S.d. $\sigma = 2.4$ minutes)

(Reg. April / May 2004)

Solution

1. Null hypothesis H_0: $\mu = 20$ minutes
2. Alternative hypothesis H_1: $\mu \neq 20$ minutes
3. Level of significance: $\alpha = 0.05$
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance. Test is T.T.T
5. Test of statistic:



$$z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$$

$$\mu = 20$$

$$\overline{x} = 20.75$$

$$n = 36$$

$$z = \frac{20.75 - 20}{2.4/\sqrt{36}} = \frac{0.75}{2.4/6} = \frac{0.75}{0.4} = 1.875$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted

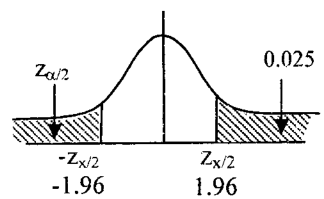
   $-1.96 < 1.875 < 1.96$ i.e. null hypothesis is accepted.

Example 3.15. It is claimed that a random sample of 49 tyres is a mean life of 15200 km. This sample was drawn from a population whose mean is 15150 km and a standard deviation of 1200km, Test the significance at 0.05 levels.

(Supple. Nov./Dec. 2005 Set 3)

Solution

1. Null hypothesis H_0: $\mu = 15150$kms
2. Alternative hypothesis H_1: $\mu \neq 15150$kms
3. Level of significance: $\alpha = 0.05$
4. Critical region: $z_{\alpha/2} = 1.96$ for α=0.05 level of significance. Test is T.T.T



5. Test of statistic:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

Population mean $\mu = 15150$ kms

Population standard deviation $\sigma = 1200$ kms

Population mean $\bar{x} = 15200$kms

Sample size n = 49

$$z = \frac{15200 - 15150}{1200/\sqrt{49}} = \frac{50}{1200/7} = \frac{50}{171.42} = 0.2911$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted

   $-1.96 < 0.2916 < 1.96$ i.e. null hypothesis is accepted

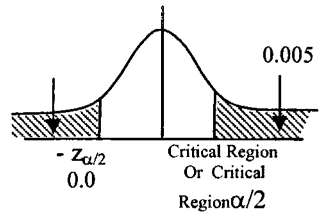
Example 3.16. A random sample from a company's very extensive files shows that the orders for a certain kind of machinery were filed, respectively in 10, 12, 19, 14, 15, 18, 11 and 13 days. Use the level of significance $\alpha = 0.01$ to test the claim that on the average such orders are field in 10.5 days. Assume normality.

(Supple. Nov. /Dec. 2004 Set 4)

Solution

1. Null hypothesis H_0: $\mu = 10.5$ days
2. Alternative hypothesis H_1: $\mu \neq 10.5$ days
3. Level of significance: $\alpha = 0.01$

4. Critical region: $z_{\alpha/2} = 2.58$ for $\alpha = 0.01$ level of significance. Test is T.T.T



5. Test of statistic:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$\bar{x} = \frac{10 + 12 + 19 + 14 + 15 + 18 + 11 + 13}{8} = \frac{112}{8} = 14$$

$$S^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

$$= \frac{(10-14)^2 + (12-14)^2 + (19-14)^2 + (14-14)^2 + (15-14)^2 + (18-14)^2 + (11-14)^2 + (13-14)^2)}{8-1}$$

$$= \frac{(-4)^2 + (-2)^2 + (5)^2 + (0)^2 + (1)^2 + (4)^2 + (-3)^2 + (1)^2}{7}$$

$$= \frac{16 + 4 + 25 + 1 + 16 + 9 + 1}{7}$$

$$= \frac{72}{7} = 10.28$$

S = 3.206

$$z = \frac{14 - 10.5}{3.206/\sqrt{8}} = \frac{3.5}{3.206/2.828}$$

$$= \frac{3.5}{1.1336} = 3.087$$

6. Decision : $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted

$-2.58 < 3.087 > 2.58$ null hypothesis is accepted

Confidence Inter

1) The mean a
respectively.
error if n=6(

[**Hint**: $E =$

2) Construct 9

[**Hint**: Confide

3) To estimate
workers in th
a standard de

a) What car
x =12.75
do the job

b) Use the g

[**Hint**: a) $E =$

b) Co

4) A random sa
the Sample

Mean is $\bar{x}$ =

[**Hint**: Confidence

EXERCISE

Confidence Interval for mean & M.E. for large sample

1) The mean and standard deviation of a population are 11,795 and 14,054 respectively. What can one assert that 99% confidence about the maximum error if n=60.

[**Hint**:. $E = z_{\alpha/2} \,.\, \sigma / \sqrt{n}$ =2.58×14054/√60 =4681.64 **Ans.**]

2) Construct 99% confidence interval for the true mean in the above problem.

[**Hint**: Confidence interval $= \overline{x} - z_{\alpha/2}\left(\sigma/\sqrt{n}\right) < \mu < \overline{x} + z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$

= (11795-4681.64, 11795+4681.64)

= (7113.35, 16476.64) **Ans.**]

3) To estimate the average time it takes to perform a task. It is timed for 40 workers in the performance of the task. Getting a mean of 12.75 minutes and a standard deviation of 2.66 minutes.

a) What can we say with 95% confidence about the maximum error if x =12.75 is used as a point estimate of the actual average time required to do the job.

b) Use the given data to construct a 95% confidence interval.

[**Hint**: a) $E = z_{\alpha/2} \,.\, \sigma / \sqrt{n}$ = 1.96× 2.66/√40=0.8244

b) Confidence interval $= \overline{x} - z_{\alpha/2}\left(\sigma/\sqrt{n}\right) < \mu < \overline{x} + z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$

= (12.75-0.8244, 12.75+0.8244)

= (11.925, 10.511) **Ans.**]

4) A random sample of size 100 is taken from a population with σ = 4.9, with the Sample

Mean is $\overline{x}$ = 20.5. Construct a 95% confidence interval for population mean μ.

[**Hint**: Confidence interval $= \overline{x} - z_{\alpha/2}\left(\sigma/\sqrt{n}\right) < \mu < \overline{x} + z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$

= {20.5-(1.96×4.9/√100), 20.5+ (1.96×4.9/√100)}

= (19.5396, 21.4604) **Ans.**]

5) A random sample of size 100 is taken from a population with σ =5.1. Given that the Sample mean is $\bar{x}$ = 21.6. Construct a 95% confidence interval for the population mean μ.

[**Hint**: confidence interval $= \bar{x} - z_{\alpha/2}\left(\sigma/\sqrt{n}\right) < \mu < \bar{x} + z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$

$= (21.6\text{-}1.96\times (5.1/\sqrt{100}), 21.6+1.96\times (5.1/\sqrt{100})$

= (20.6 22.6) **Ans.**]

6). Measurements of the weights of a random sample of 200 mangoes which are purchased from a market, with mean 0.850 and standard deviation of 0.032. Find 95% confidence limits for the mean weight of all the mangoes.

[**Hint**: confidence limits = $\bar{x} - z_{\alpha/2}\left(\sigma/\sqrt{n}\right) < \mu < \bar{x} + z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$

$= (0.850\text{-}1.36\times (0.032/\sqrt{200}), 0.850+1.90\times 0.032/\sqrt{200}))$

= (0.8455, 0.8544) **Ans.**]

Hypothesis Concerning one mean Large Sample

7) According to the norms established for a mechanical aptitude test, persons who are 18 years old have an average height of 73.2 with a standard deviation of 8.6. If 45 Randomly selected persons of that age averaged 76.7. Test the null hypothesis μ= 73.2 against the alternative hypothesis μ>73.2

[**Hint**: N.H: μ=73.2

A.H: μ>73.2

Test statistic $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$= (76.7\text{-}73.2)/ (8.6/\sqrt{45})$

= 2.73

Z=2.73>$Z\alpha$=2.33

Ho is rejected **Ans.**]

8) It is claimed that a random sample of 50 pens with a mean life of 96 Hrs which is drawn from a population of pens which has a mean life of 90Hrs and a standard deviation 30 Hrs. test validity of this claim.

[Hint: N.H: μ= 96hrs.

A.H: $\mu \neq$ 96

L.

Te

9) A sample o is 100. The that Populat

[Hint: N.H

A.H

Test stastistic

Z=

Confidence Int

= (

(Hypothesis conce

10) Two sample

weights and the Differen

[**Hint**: N.H Ho:

A.H H1

L.

A.H: $\mu \neq 96$hrs.

L.O.S. $\alpha = 0.0$

Test statistic $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ $\qquad$ $Z_{\alpha/2} = 1.96$

$= (96-90)/\ (30/\sqrt{50})$ $\qquad$ $Z = 1.414 < Z\alpha/2 = 1.96$

$= 1.414$ $\qquad$ N.H is accepted **Ans.**]

9) A sample of 400 printers is taken from a population whose standard deviation is 100. The mean life time of the cartage of the printer is 40. Test the claim that Population mean is 30. Also find 95% confidence interval.

[Hint: N.H Ho $\mu = 30$

A.H H_1 $\mu \neq 30$

Test stastistic $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}} = (40-30)/100/\sqrt{400} = 2$

$Z = 2 > Z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ L.O.S Ho is rejected]

Confidence Interval $= \bar{x} - z_{\alpha/2}\left(\sigma / \sqrt{n}\right) < \mu < \bar{x} + z_{\alpha/2}\left(\sigma / \sqrt{n}\right)$

$= (40 - 1.96 \times 100/\sqrt{400}\ ,\ \ 40 + 1.96 \times 100/\sqrt{400}) = (30.2\ ,\ \ 49.8)$]

(Hypothesis concerning two means large samples)

10) Two samples of the students are selected from two different departments their weights and standard deviation are given as follows. Test the significance of the Difference between the means.

	Mean	S.D	Size of the sample
CSE Dept	55	15	70
IT Dept	70	20	90

[**Hint**: N.H Ho: $\mu_1 = \mu_2$

A.H H1: $\mu_1 \neq \mu_2$

L.O.S = 0.05

Test statistic $z = \dfrac{\bar{x}-\bar{y}}{\sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}}$ = (55-70)/ √ (15^2/70 + 20^2/90)

= -5.4202

-5.4202 < Z_α=1.96, Ho is accepted]

(Large Sample, Difference between the mean)

11) Two sample sizes are 100 and 50. The standard deviations are 5 and 8mean of the sample are (50, 40). Determine the standard error of the difference between the means and also find the confidence interval at 0.05 levels.

[Hint: S.E = $\sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}$) = √ (5^2/100) + (8^2/ 50)

Confidence interval $=(\bar{x}-\bar{y}) \pm z_{\alpha/2} \times \sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}$

= 50-40 ±1.96√25/100+64/50

= (12.429, 7.575) Ans.]

OBJECTIVE TYPE QUESTIONS

1. A (1 – α)100% confidence internal for μ is given by
   (a) $\bar{x}-z_\alpha\ \sigma/\sqrt{n} < \mu < \bar{x}+z_\alpha\ \sigma/\sqrt{n}$
   (b) $\bar{x}-z_{\alpha/4}\ \sigma/\sqrt{n} < \mu < \bar{x}+z_{\alpha/4}\ \sigma/\sqrt{n}$
   (c) $\bar{x}-z_{\alpha/2}\ \sigma/\sqrt{n} < \mu < \bar{x}+z_{\alpha/2}\ \sigma/\sqrt{n}$
   (d) None of these

Ans. (c)

2. The maximum error of estimate
   (a) $E = z_{\alpha/2}.\sigma/\sqrt{N}$ (b) $E = z_{\alpha/2}.\sigma/\sqrt{n}$
   (c) $E = z_{\alpha/2}.\sigma^2/\sqrt{n}$ (d) None of these

Ans. (b)

3. The formula for
   (a) $n = \left[\dfrac{z_\alpha.\sigma}{E}\right]^2$
   (c) $n = \left[\dfrac{z_{\alpha/2}.\sigma}{E}\right.$

Ans. (c)

4. A Sample of si
   probability 0.95,
   (a) 0.87

Ans. (d)

5. If the standard d
   99% confidence.
   (a) 900

Ans. (a)

6. If the sample siz
   will be the 99% c
   (a) (150, 170)
   (c) (155.03, 171.

Ans. ()

7. If the maximum
   size?
   (a) 1540

Ans. (c)

8. If the maximum
   variance is?
   (a) $(65.02)^2$

Ans. (b)

9. Two samples wh
   the σ_{x-y}^2
   (a) 70

Ans. (c)

3. The formula for the sample size

(a) $n = \left[\dfrac{z_{\alpha}.\sigma}{E}\right]^2$ (b) $n = \left[\dfrac{z^{\alpha/2}.\sigma}{E}\right]^2$

(c) $n = \left[\dfrac{z_{\alpha/2}.\sigma}{E}\right]^2$ (d) $n = \left[\dfrac{z_{\alpha/2}.\sigma}{E^2}\right]^2$

Ans. (c)

; and 8mean
lifference
levels.

4. A Sample of size 100 is taken whose standard deviation is 5.With the probability 0.95, what is the maximum error?

(a) 0.87 (b) 0.6 (c) 1 (d) 0.98

Ans. (d)

5. If the standard deviation of a sample is 20 and maximum error is 1.72 with 99% confidence. What will be the sample size?

(a) 900 (b) 800 (c) 200 (d) None of the above

Ans. (a)

6. If the sample size is 64, mean is 165 with the standard deviation is 25, what will be the 99% confidence interval for mean.

(a) (150, 170) (b) (156.94, 173.06)

(c) (155.03, 171.05) (d) None of the above

Ans. ()

7. If the maximum error is 0.06, σ =1, with 99% confidence, what is the sample size?

(a) 1540 (b) 840 (c) 1849 (d) None of the above

Ans. (c)

8. If the maximum error with 90% confidence is 2.8 and sample size is 750 then variance is?

(a) $(65.02)^2$ (b) $(46.61)^2$ (c) $(50.53)^2$ (d) None of the above

Ans. (b)

9. Two samples whose standard deviations 8 & 5 and sample sizes are 100 & 64 the $\sigma_{x-y}^{\ 2}$

(a) 70 (b) 105 (c) 103 (d) 101

Ans. (c)

10. If the Standard deviation of population is 5, n=169, mean=50 then 99% confidence interval for μ is

(a) (50, 52) (b) (48, 50) (c) (51, 53) (d) (49, 51)

Ans. (d)

11. Sample size is given by

(a) $n = \left[\frac{z_{\alpha/2}\cdot}{E}\right]^2$ (b) $n = \left[\frac{E}{z_{\alpha/2}}\right]^2$

(c) $n = \left[\frac{z_{\alpha/2}.\sigma}{E}\right]^2$ (d) None of the above

Ans. (c)

12. Maximum error is given by

(a) $E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ (b) $E = t_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ (c) $E = \frac{\sigma}{\sqrt{n}}$ (d) None of the above

Ans. (a)

13. In a random sample of 400 items it is found that 231 are damage because of quality of production. Construct a 99% confidence interval for the corresponding true proportion.

(a) (0.4, 0.7) (b) (0.514, 0.642) (c) (0.5, 0.6) (d) (0.642, 0.751)

Ans. (b)

14. If p = 0.578, q=0.42. n = 400 find maximum error with 95% confidence for true proportion

(a) 0.078 (b) 0.048 (c) 0.058 (d) none of these

Ans. (b)

15. Among 100 people in a state 10 are found to be chapatti eaters. Construct 99% confidence interval for the true proportion.

(a) (0.579, 0.1258) (b) (0.0742, 0.1258)

(c) (0.0742, 0.1520) (d) none of these

Ans. (b)

16. A random samp
to be bad. Wha
of proportion.
(a) 0.03

Ans. (c)

17. In a sample of
you say about t
(a) 0.0582

Ans. (b)

18. If we can assert
the size of the s
(a) 71

Ans. (a)

19. If the maximum
then the varianc

(a) 2.166

Ans. (c)

20. Among 100 fac
confidence, the
(a) 0.059

Ans. (c)

21. A random samp
proportion with
(a) 0.0329

Ans. (b)

22. If p = 0.5 and
confidence is
(a) 0.567

Ans. (d)

16. A random sample of 100 bananas were taken and out of which 15 were found to be bad. What can we say with 95% confidence about the maximum error of proportion.

    (a) 0.03 (b) 0.05 (c) 0.07 (d) none of these

Ans. (c)

17. In a sample of 700 people in Andhra Pradesh 300 are rice eaters. What can you say about the maximum error with 99% confidence?

    (a) 0.0582 (b) 0.0482 (c) 0.0182 (d) none of these

Ans. (b)

18. If we can assert with 95% that the maximum error is 0.07 and p is 0.1. Find the size of the sample

    (a) 71 (b) 50 (c) 81 (d) none of these

Ans. (a)

19. If the maximum error with 99% probability is 0.25 and sample size n=500 then the variance of the population is

    (a) 2.166 (b) 3.205 (c) 4.694 (d) none of these

Ans. (c)

20. Among 100 faculty members in college 80 people use cell phone. With 95% confidence, the maximum error for true proportion is

    (a) 0.059 (b) 0.085 (c) 0.0784 (d) none of these

Ans. (c)

21. A random sample of 500 products, 100 are defective items. Standard error of proportion with 99% is

    (a) 0.0329 (b) 0.0461 (c) 0.165 (d) 0.521

Ans. (b)

22. If p = 0.5 and the sample size is 70 then the maximum error with 99% confidence is

    (a) 0.567 (b) 0.1212 (c) 0.25 (d) 0.1541

Ans. (d)

23. If q = 0.3 and the sample size is 140 then the maximum error with 95% confidence is

(a) 0.0459 (b) 0.0444 (c) 0.0559 (d)0.0759

Ans. (d)

24. For the level of significance 1% & 5 % respectively what are the critical values of Z_α for two tailed test

(a) 2.33, 1.645 (b) -2.33, -1.645 (c)2.58, 1.96 (d) None of the above

Ans. (c)

T

4.1 TEST OF I
(SMALL SAM

Working Proc

1. Null hypo
2. Alternativ
3. Level of si
4. Test of sta
Ca

a. If P ≠

P =

P =

b. If P >

P =

UNIT-4

TESTING OF HYPOTHESIS-II

"Mind it self is the form of all."

4.1 TEST OF HYPOTHESIS CONCERNING ONE PROPORTION (SMALL SAMPLES)

(Nov. 2009 set 4)

Working Procedure

1. Null hypothesis H_0: $P = P_0$
2. Alternative hypothesis H_1: $P = P_0$ or $P > P_0$ or $P < P_0$
3. Level of significance: α
4. Test of statistic

   Calculate P – value

   a. If $P \neq P_0$ then

   $$P = 2P\,(X \leq x) \text{ if } x < n.P_0$$

   $$P = 2P\,(X \geq x) \text{ if } x > nP_0$$

   b. If $P > P_0$ then

   $$P = P\,(X \geq x)$$

c. If $P < P_0$ then

$$P = P(X \le x)$$

Where x is the number of success.

5. Decision
If $P > \alpha$ then null hypothesis is accepted otherwise $P \le \alpha$ null hypothesis is rejected

4.2 TEST OF HYPOTHESIS CONCERNING ONE PROPORTION (LARGE SAMPLE)

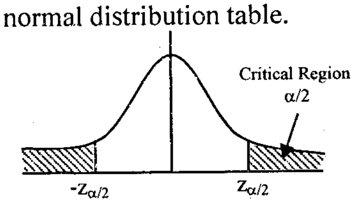
Working procedure

(Nov.2009 set 4)

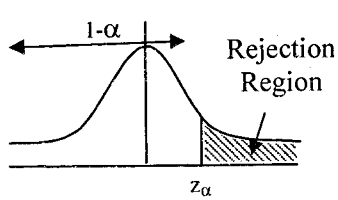
For the large sample $n \ge 30$. Formulate

1. Null hypothesis H_0: $P = P_0$
2. Alternative hypothesis H_1: $P \ne P_0$ or $P > P_0$ or $P < P_0$
3. Level of significance: α
4. Critical Region:

a. If $P \ne P_0$ test is two tail test, for given α, critical values are $-z_{\alpha/2}$ and $z_{\alpha/2}$ from the normal distribution table.



b. If $P > P_0$ test is right one tail test, for given α, critical value is z_α from the Normal distribution table.



c. If $P < P_0$ test is left one tail test, for given α,critical value is $-z_\alpha$ from the Normal distribution table.

5. Test of statistic

$$z = \frac{P - p}{\sqrt{Pq/n}}$$

Where $P = $ …

6. Decision

a. If $-z_{\alpha/2} < z$ …

Hypothesis is …

b. $z_\alpha > z$

Hypothesis is …

c. $Z > -Z_\alpha$

Hypothesis is …

4.3 TEST OF HYP… (LARGE SAMPL…

Working Procedure

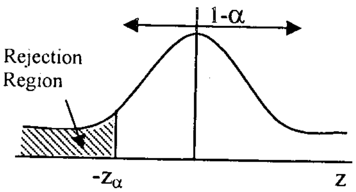
1. Null hypot…

Where P_1 = …

2. Alternative …
3. Level of sig…
4. Critical Reg…

a. If $P_1 - P_2 \ne 0$ test is …

$z_{\alpha/2}$ from the norma…



5. Test of statistic

$$z = \frac{P - p}{\sqrt{Pq / n}}$$

Where $P = \frac{x}{n}$

6. Decision

   a. If $-z_{\alpha/2} < z < z_{\alpha/2}$

   Hypothesis is accepted for two tail test otherwise rejected

   b. $z_{\alpha} > z$

   Hypothesis is accepted for right one tail test otherwise rejected

   c. $Z > -Z_{\alpha}$

   Hypothesis is accepted for left one tail test otherwise rejected

4.3 TEST OF HYPOTHESIS CONCERNING TWO PROPORTIONS (LARGE SAMPLES)

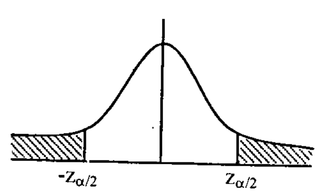
Working Procedure

1. Null hypothesis $H_0 : P_1 = P_2$ or $P_1 - P_2 = 0$

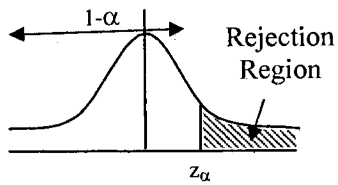
   Where $P_1 = \frac{x_1}{n_1}$ *and* $P_2 = \frac{x_2}{n_2}$

2. Alternative hypothesis H_1: $P_1 - P_2 \neq 0$ or $P_1 > P_2$ or $P_1 < P_2$
3. Level of significance: α
4. Critical Region:

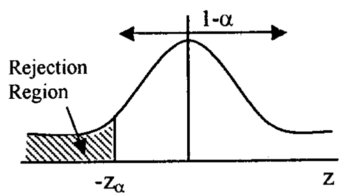
a. If $P_1 - P_2 \neq 0$ test is two tail test, for given α, critical values are $-z_{\alpha/2}$ and $z_{\alpha/2}$ from the normal distribution table.



b. If $P_1 > P_2$ test is right one tail test, for given α,critical value is z_α from the normal distribution table.



c. If $P_1 < P_2$ test is left one tail test, for given α, critical value is $-z_\alpha$ from the normal distribution table.



5. Test of statistic

$$z = \frac{P_1 - P_2}{\sqrt{P(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$\text{Where } P_1 = \frac{x_1}{n_1},\ P_2 = \frac{x_2}{n_2}$$

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$$

n_1 and n_2 sizes of two samples, which are drawn from the two distinct populations.

6. Decision

a. If $-z_{\alpha/2} < z < z_{\alpha/2}$
   Hypothesis is accepted for two tail test otherwise rejected

b. $z_\alpha > z$
   Hypothesis is accepted otherwise rejected for right one tail test

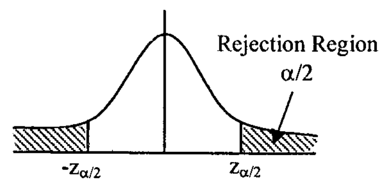
c. $z > -z_\alpha$
   Hypothesis is accepted otherwise rejected for left one tail test.

4.4 TEST OF HYPOTHESIS CONCERNING DIFFERENCE BETWEEN PROPORTIONS (LARGE SAMPLES)

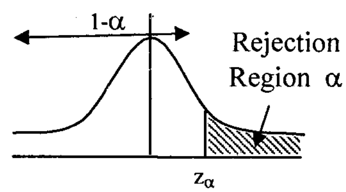
Working procedure

1. Null hypothesis H_0: $P_1 - P_2 = C$ (Constant)
2. Alternative hypothesis H_1: $P_1 - P_2 \neq C$ or $P_1 - P_2 > C$ or $P_1 - P_2 < C$
3. Level of significance: α
4. Critical Region:

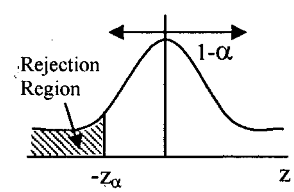
a. If $P_1 - P_2 \neq C$ test is two tail test, for given α, critical values are $-z_{\alpha/2}$ and $z_{\alpha/2}$ from the normal distribution table.



b. If $P_1 - P_2 > C$ test is right one tail test, for given α, critical value is z_α from the normal distribution table.



c. If $P_1 - P_2 < C$ test is left one tail test, for given α, critical value is $-z_\alpha$ from the normal distribution table.



5. Test of statistic

$$z = \frac{(P_1 - P_2) - C(Constnat)}{\sqrt{\frac{P_1(1-P_1)}{n_1} + \frac{P_2(1-P_2)}{n_2}}}$$

Where $P_1 = \dfrac{x_1}{n_1}$, $P_2 = \dfrac{x_2}{n_2}$

6. Decision

   a. If $-z_{\alpha/2} < z < z_{\alpha/2}$

   Hypothesis is accepted for two tail test otherwise rejected

   b. $z_{\alpha} > z$

   Hypothesis is accepted otherwise rejected for right one tail test

   c. $Z > -z_{\alpha}$

   Hypothesis is accepted otherwise rejected for left one tail test.

1. Confidence interval for proportions (large sample)

$$\frac{x}{n} - z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}} < P < \frac{x}{n} + z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}$$

Where n $\rightarrow$ sample size (Large)
x $\rightarrow$ Number of successes

2. Maximum error is given by

$$M.E. = z_{\alpha/2}\sqrt{\frac{P(1-P)}{n}}$$

3. Sample size :

   a. If P is given

$$n = P(1-P)\left(\frac{z_{\alpha/2}}{E}\right)^2$$

   b. If P is not given the sample size

$$n = \left(\frac{1}{4}\right)\left(\frac{z_{\alpha/2}}{E}\right)^2$$

SOLVED EXAMPLES

Example 4.1. 400 articles from a factory are examined and 3% are found to be defective. Construct 95% confidence interval. **(Nov. 2006 Set 3)**

Solution

Given that $z_{\alpha/2} = 1.96$ for confidence level 95%
Sample size n = 400
P = 0.03 or 3%

Confidence

Confidence

$(0.03)-1.$

$(0.03)-$

$(0$

Example 4.2. In
with explosives i
explosive when
Find a 95% confi

Solution
(1-α) 100%

$\frac{x}{n} - z_{\alpha/2}$

Given that x
n = 200
$z_{\alpha/2} = 1.96$

$\frac{174}{200} - 1.$

Confidence interval =

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Confidence internal for proportion

$$p - z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} < p < p + z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

$$(0.03) - 1.96\sqrt{\frac{(0.03)(1-0.03)}{400}} < p < (0.03) + 1.96\sqrt{\frac{(0.03)(1-0.03)}{400}}$$

$$(0.03) - 1.96\sqrt{\frac{(0.03)(0.97)}{400}} < p < (0.03) + 1.96\sqrt{\frac{(0.03)(0.97)}{400}}$$

$$(0.03) - 1.96\sqrt{\frac{0.0291}{400}} < p < (0.03) + 1.96\sqrt{\frac{0.0291}{400}}$$

$$0.03 - (0.0167) < p < 0.03 + 0.0167$$

$$0.0132 < p < 0.0467$$

Confidence interval (0.0132, 0.0467)

Example 4.2. In a study designed to investigate whether certain detonators used with explosives in coal mining meet the requirement that at least 90% will ignite the explosive when charged, it is found that 174 of 200 detonators function properly. Find a 95% confidence internal for the tune proportion?

(Supple. Feb. 2007 Set 1, Set 2)

Solution

(1-α) 100% confidence interval for true proportion

$$\frac{x}{n} - z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}} < p < \frac{x}{n} + z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}$$

Given that x = 174

n = 200

$z_{\alpha/2} = 1.96$ For $\alpha = 0.95$

$$\frac{174}{200} - 1.96\sqrt{\frac{\frac{174}{200}\left(1-\frac{174}{200}\right)}{200}} < p < \frac{174}{200} + 1.96\sqrt{\frac{\frac{174}{200}\left(1-\frac{174}{200}\right)}{200}}$$

$$0.87-1.96\sqrt{\frac{(0.87)(1-0.87)}{200}}<p<0.87+1.96\sqrt{\frac{(0.87)(1-0.8)}{200}}$$

$$0.87-1.96\sqrt{\frac{0.1131}{200}}<p<0.87+1.96\sqrt{\frac{0.1131}{200}}$$

$0.87 - 0.0466 < p < 0.87 + 0.0466$

$0.8233 < 9 < 0.9166$ Ans.

Example 4.3. If we can assert with 95% that the maximum error is 0.5 and p = 0.2 find the ample size. **(Supple. Nov. / Dec. 2005)**

Solution

We know that when p is known, the sample size n is given by

$$n = p(1-p)\left(\frac{z_{\alpha/2}}{E}\right)^2$$

Given that

Maximum error E = 0.5

p = 0.2

n =?

$$1-\alpha = 0.95$$
$$1-0.95 = \alpha$$
$$\alpha = 0.05$$
$$z_{\alpha/2} = z_{0.05/2} = z_{0.025} = 1.96$$
$$n = (0.2)(1-0.2)\left[\frac{1.96}{0.5}\right]^2$$
$$= (0.2)(0.8)(3.92)^2$$

$n = 2.458$ **Ans.**

Example 4.4. A manufacturer of electric bulbs claims that the percentage defective in his product doesn't exceed 6. A sample of 40 bulbs is found to contain 5 defectives would you consider the claim justified?

Solution

1. Null hypothesis H_0: p = .06
2. Alternative hypothesis H_1: p < .06
3. Level of significance α= 0.05
4. Critical region: $-z_\alpha = -1.645$ for α = 0.05 L.O.S.

   Test is L.O.T.T.

5. Test of :

$z = \frac{P}{\sqrt{p}}$

$p = \frac{x}{n} =$

$= \frac{5}{4(}$

$z =$

$=$

$= 1.7$

6. Decision

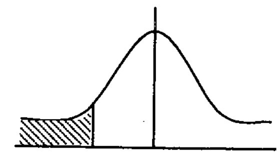
- 1.96 <

Example 4.5. A m in his product doe defectives would yo

Solution

1. Null hyp
2. Alternati
3. Level of
4. Critical r

   Test is L.



$-1.645 = -z_\alpha$

5. Test of statistic or computation

$$z = \frac{P - p}{\sqrt{pq/n}}$$

$$p = \frac{x}{n} = \text{Proportion of successes in the sample}$$

$$= \frac{5}{40} = 0.125$$

$$z = \frac{0.125 - 0.06}{\sqrt{\frac{(0.06)(1-0.06)}{40}}} = \frac{0.065}{\sqrt{\frac{(0.06)(0.94)}{40}}}$$

$$= \frac{0.065}{\sqrt{\frac{0.0564}{40}}} = \frac{0.065}{\sqrt{0.00141}} = \frac{0.065}{0.0375}$$

$$= 1.7333$$

6. Decision : $-z_{\alpha/2} < Z$, accepted

   $-1.96 < 1.73$ null hypothesis is accepted

Example 4.5. A manufacturer of electric bulbs claims that the percentage defective in his product doesn't exceed 6. A sample of 10 bulbs is found to contain 5 defectives would you consider the claim justified?

(Feb. 2007 Set 4)

Solution

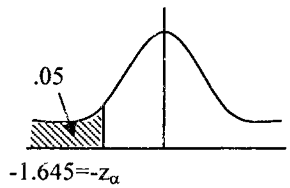
1. Null hypothesis H_0: $p = 0.06$

2. Alternative hypothesis H_1: $p < 0.06$

3. Level of significance $\alpha = 0.05$

4. Critical region: $-z_\alpha = -1.645$ for $\alpha = 0.05$ level of significance

   Test is L.O.T.T.



5. Test of statistic or computation

$P = P\,(X \le x \text{ when } P = p_0)$

$p = 5,\ np_0 = 10 \times 0.06 = 0.6$

$X = 5,\ np_0 = 10 \times 0.06 = 0.6$

$5 = X > np_0 = 0.6$

$P = p(X \le 5 \text{ when } p = 0.06)$

$$= \sum_{X=0}^{5} b(X;10,\ 0.06)$$

$$= \sum_{x=0}^{5} 10_{C_x} (0.06)^x (1-0.06)^{10-x}$$

$$= \sum_{x=0}^{5} 10_{C_x} (0.06)^x (0.94)^{10-x}$$

$$= 10_{C_0}(0.06)^0(0.94)^{10-0} + 10_{C_1}(0.06)^1(0.94)^{10-1} + 10_{C_2}(0.06)^2(0.94)^{10-2}$$
$$+ 10_{C_3}(0.06)^3(0.94)^{10-3} + 10_{C_4}(0.06)^4(0.94)^{10-4}$$
$$+ 10_{C_5}(0.06)^5(0.94)^{10-5}$$

$$= \frac{\angle 10}{\angle 10\ \angle 10-0}(0.5386) + \frac{\angle 10}{\angle 1\ \angle 10-1}(0.06)(0.5729) + \frac{\angle 10}{\angle 2\ \angle 10-2}(0.0036)(0.6095)$$
$$+ \frac{\angle 10}{\angle 3\ \angle 10-3}(0.000216)\times(0.6484) + \frac{\angle 10}{\angle 4\ \angle 10-4}(0.000012)(0.6898)$$
$$+ \frac{\angle 10}{\angle 5\ \angle 10-5}(0.0000077)\times(0.73390)$$

$= (0.5386) + 10 \times (0.034374) + 45 \times (0.02194) + 120 \times (0.000140)$
$\quad + 210 \times (0.0000083) + 252 \times (0.000000565)$
$= 0.5386 + 0.34374 + 0.9873 + 0.0168 + 0.00174 + 0.000142$
$= 1.888$

6. Decision:
Reject
$1.88 \le$ (
Hypoth

Example 4.6. In
with explosives in
explosive when ch
Test null hypothes
level of significanc

Solution

1. Null hy
2. Alternat
3. Level o
4. Critical

Test is I

5. Test of s

$z = \frac{P}{\sqrt{p}}$

$P = \frac{17}{20}$

$z =$

$= \frac{-0.}{\sqrt{0.0}}$

$= \frac{-0.0}{0.021}$

$= -1.41$

6. Decision

$-1.41 >-$

6. Decision:

   Reject N.H H_0 if $p \le \alpha$

   $1.88 \le 0.05$

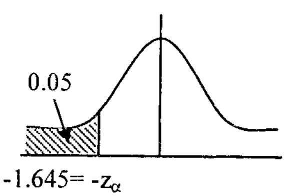
   Hypothesis is rejected

Example 4.6. In a study designed to investigate whether certain detonators used with explosives in coal mining meet the requirement that at least 90% will ignite the explosive when charged, it is found that 174 of 200 detonators function properly. Test null hypothesis P = 0.90 against the alternative hypothesis p< 0.90 at the 0.05 level of significance. **(JNTU 2001)**

Solution

1. Null hypothesis H_0: p = 0.90
2. Alternative hypothesis H_1: p < .90
3. Level of significance $\alpha = 0.05$
4. Critical region: $-z_\alpha = -1.645$ for $\alpha = 0.05$ level of significance

   Test is L.O.T.T.



5. Test of statistic or computation

$$z = \frac{P-p}{\sqrt{pq/n}}$$

$$P = \frac{174}{200} = 0.87$$

$$z = \frac{0.87-0.90}{\sqrt{\frac{(0.90)\,(1-0.90)}{200}}} = \frac{-0.03}{\sqrt{\frac{(0.90)(0.1)}{200}}}$$

$$= \frac{-0.03}{\sqrt{0.00045}}$$

$$= \frac{-0.03}{0.0212}$$

$$= -1.41$$

6. Decision: If $P < p_0$ and $z < -z_\alpha$ then null hypothesis rejected

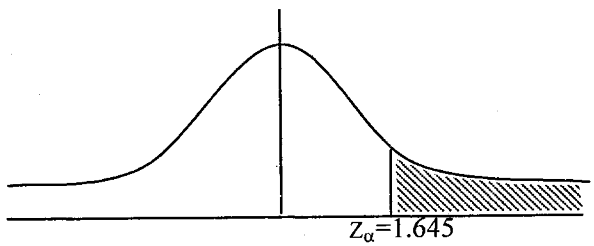
   - 1.41 >– 1.645 then null hypothesis is accepted

Example 4.7 In a random sample of 125 cola drinkers, 68 said they prefer thumpsup to Pepsi. Test the null hypothesis P = 0.5 against the alternative hypothesis p< 0.5 **(Supple. Nov. 2008)**

Solution

1. Null hypothesis H_0: p = 0.5
2. Alternative hypothesis H_1: p >0.5
3. Level of significance $\alpha = 0.05$
4. Critical region: $z_\alpha = 1.645$ for $\alpha = 0.05$ level of significance

   Test is R.O.T.T.



5. Test of statistic or computation

$$z = \frac{P - p}{\sqrt{pq/n}}$$

$$P = \frac{x}{n} = \frac{68}{125} = 0.544$$

$$z = \frac{0.544 - 0.5}{\sqrt{\frac{(0.5)\,(1 - 0.5)}{125}}} = \frac{0.044}{\sqrt{\frac{(0.5)(0.5)}{125}}}$$

$$= \frac{0.044}{\sqrt{0.002}} = \frac{0.044}{0.04472} = 0.9838$$

6. Decision: If $z < z_\alpha$ then null hypothesis is accepted

   0.9838 < 1.645 then null hypothesis is accepted **Ans.**

Example 4.8. Among 900 people in a state 90 are found to be chapatti eaters. Construct 99% confidence interval for the true proportion.

(Reg. April / May 2005 Set 1)
(Supple. Feb. 2007 Set 3)
(Nov. 2006 Set 1)

Solution

The confidence interval for P when n is large

$$\frac{x}{n} - z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}} < p < \frac{x}{n} + z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}$$

$z_{\alpha/2} = 2.575$ for 99 % level of significance

$$\frac{90}{900} - 2.575\sqrt{\frac{\frac{90}{900}\left(1-\frac{90}{900}\right)}{900}} < p < \frac{90}{900} + 2.575\sqrt{\frac{\frac{90}{900}\left(1-\frac{90}{900}\right)}{900}}$$

$$0.1 - 2.575\sqrt{\frac{(0.1)(0.9)}{900}} < p < 0.1 + 2.575\sqrt{\frac{(0.1)(0.9)}{900}}$$

$$0.1 - 2.575 \times 0.01 < p < 0.1 + 2.575 \times 0.01$$

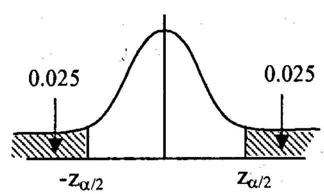
$$0.1 - 0.02575 < p < 0.1 + 0.02575$$

$$0.07425 < p < 0.12575$$ **Ans.**

Example 4.9. Experiences had shown that 20% of a manufactured product is of the top quality. In one day production of 400 articles only 50 are of top quality. Test the hypothesis at 0.05 levels. **(Reg. April / May 2005 Set 3)**

Solution

1. Null hypothesis H_0: p = 0.20
2. Alternative hypothesis H_1: $p \neq 0.20$
3. Level of significance $\alpha = 0.05$
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance



Test of statistic or computation

$$z = \frac{P - p}{\sqrt{pq/n}}$$

$$p = \frac{x}{n} = \frac{50}{400} = 0.125$$

$$z = \frac{0.125 - 0.20}{\sqrt{\dfrac{0.20(1-0.20)}{400}}} = \frac{-0.075}{0.02}$$

$= -3.75$

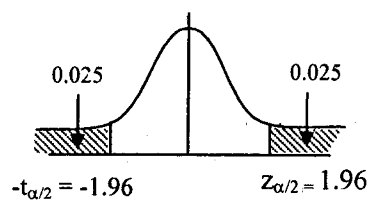
6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted.
   $-1.96 > -3.75 < 1.96$ null hypothesis is rejected

Example 4.10. A manufacturer of electronic equipment subjects samples of two competing brands of transistors to an accelerated performance test. If 45 of 180 transistors of the first kind and 34 of 120 transistors of the second kind fail the test, what can he conclude at the level of significance $\alpha = 0.05$ about the difference between the corresponding sample proportions?

(Reg. April / May 2004)

Solution

1. Null hypothesis H_0: $P_1 = P_2$
2. Alternative hypothesis H_1: $P_1 \neq P_2$
3. Level of significance $\alpha = 0.05$
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance. Test is T.T.T



5. Test of statistic:

$$z = \frac{\dfrac{x_1}{n_1} - \dfrac{x_2}{n_2}}{\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$$

Where $\hat{p} = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \dfrac{x_1 + x_2}{n_1 + n_2}$

$n_1 = 180$, $x_1 = 45$
$n_2 = 120$, $x_2 = 34$

$$\hat{p} = \frac{45+34}{180+120} = \frac{79}{300} = 0.2633$$

$$z = \frac{\dfrac{45}{180} - \dfrac{34}{120}}{\sqrt{(0.2633)(1-0.2633)\left(\dfrac{1}{180} + \dfrac{1}{120}\right)}}$$

:

=

Z =

6. Decision:
   - 1.96 < -
   There is

Example 4.11. A
Random samples of
at .05 level.

Solution

1. Null hyp
2. Alternati
3. Level of
4. Critical r
   level of s
5. Test of st
6. Decision:
   - 1.96 < 1

$$= \frac{0.25 - 0.28}{\sqrt{(0.2633)(0.7367)(0.00555 + 0.00833)}}$$

$$= \frac{-0.03}{\sqrt{0.00269}}$$

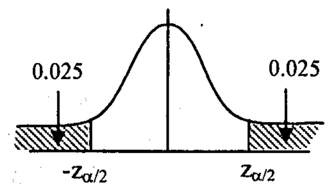
$$Z = -0.57915$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted
   $-1.96 < -0.57915 < 1.96$ null hypothesis is accepted
   There is no significant difference between two brands of transistors

Example 4.11. A manufacturer claims that only 4% of his products are defective. Random samples of 500 were taken among which 100 defective test the hypothesis at .05 level. **(Supple. Nov. / Dec. 2005 Set 2)**

Solution

1. Null hypothesis H_0: $p = 0.04$
2. Alternative hypothesis H_1: $p \neq 0.04$
3. Level of significance $\alpha = 0.05$
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance. Test is T.T.T



5. Test of statistic:

$$z = \frac{P - p}{\sqrt{pq/n}}$$

$$p = \frac{x}{n} = \frac{100}{500} = 0.2$$

$$z = \frac{0.2 - 0.04}{\sqrt{(0.04)(1 - 0.04)/500}}$$

$$= \frac{0.16}{\sqrt{\frac{(0.04)(0.96)}{500}}}$$

$$= \frac{0.16}{0.00876}$$

$$= 18.26$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted
   $-1.96 < 18.26 > 1.96$ null hypothesis is rejected

Example 4.12. On the basis of their total scores, 200 candidates of a civil service examination are divided into two groups, the upper 30% and the remaining 70% consider the first question of the examination. Among the first group, 40 had the correct answer, whereas among the second group, 80 had the correct answer. On the basis of these result, can one conclude that the first question is no good at discriminating ability of the type being examined here?

(Reg. April / May 2004)

Solution

30% of the 200 candidates are 60 candidates

70% of the 200 candidates are 140 candidates

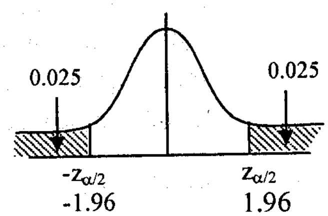
In the first group 40 had the correct answer

$$p_1 = \frac{x_1}{n_1} = \frac{40}{60} = 0.666$$

In the second group 80 had the correct answer

$$\frac{x_2}{n_2} = p_2 = \frac{80}{140} = 0.571$$

1. Null hypothesis H_0: $p_1 = p_2$
2. Alternative hypothesis H_1: $p_1 \neq p_2$
3. Level of significance $\alpha = 0.05$
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance.



5. Test of statistic

$$z = \frac{\dfrac{x_1}{n_1} - \dfrac{x_2}{n_2}}{\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$$

Where

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{40+80}{60+140} = \frac{120}{200} = 0.6$$

$$z = \frac{\dfrac{40}{60} - \dfrac{80}{140}}{\sqrt{(0.6)(1-0.6)\left(\dfrac{1}{60} + \dfrac{1}{140}\right)}}$$

6. Decision:

$- 1.96 < 1.$

Example 4.13. If 8

99% confidence limit

Solution

The confidence i

$\frac{x}{n} - z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1\right.}{}}$

$\frac{59}{80} - 2.58\sqrt{\frac{\frac{59}{80}(}{}}$

$0.7375 - 2.58\sqrt{\frac{0.73}{}}$

$0.7375 - 2.58 \times ($

$0.7375 - 0.1269$

0.6105

Example 4.14. Wha

unknown proportion

confidence?

Solution

Sample size

$n = \frac{1}{4}\left[\frac{z_{\alpha/2}}{E}\right]^2$

Given that maxim

$z_{\alpha/2} = 1.96$ for $\alpha =$

$$= \frac{0.666 - 0.5714}{\sqrt{(0.6)(0.4)(0.0166 + 0.00714)}}$$

$$= \frac{0.0946}{0.07548}$$

$$Z = 1.2533$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted

   $-1.96 < 1.2533 < 1.96$ null hypothesis is accepted

Example 4.13. If 80 patients are treated with an antibiotic 59 got cured. Find a 99% confidence limits to the true proportion of cure?

(Supple. Nov. /Dec. 2001 Set 1)

Solution

The confidence interval for p when n is large

$$\frac{x}{n} - z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}} < p < \frac{x}{n} + z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}$$

$$\frac{59}{80} - 2.58\sqrt{\frac{\frac{59}{80}\left(1-\frac{59}{80}\right)}{80}} < p < \frac{59}{80} + 2.58\sqrt{\frac{\frac{59}{80}\left(1-\frac{59}{80}\right)}{80}}$$

$$0.7375 - 2.58\sqrt{\frac{0.7375(1-0.7375)}{80}} < p < 0.7375 + 2.58\sqrt{\frac{0.7375(1-0.7375)}{80}}$$

$$0.7375 - 2.58 \times 0.04919 < p < 0.7375 + 2.58 \times 0.04919$$

$$0.7375 - 0.1269 < p < 0.7375 + 0.1269$$

$$0.6105 < p < 0.8644$$

Ans.

Example 4.14. What is the size of the smallest sample required to estimate an unknown proportion to within a maximum error of 0.06 with at least 5% confidence?

(Supple. Nov. / Dec. 2004 Set 3)

Solution

Sample size

$$n = \frac{1}{4}\left[\frac{z_{\alpha/2}}{E}\right]^2 \quad \text{When p is unknown}$$

Given that maximum error E = 0.06

$z_{\alpha/2} = 1.96$ for $\alpha = 5\%$

$$n = \frac{1}{4}\left[\frac{1.96}{0.06}\right]^2 = \frac{1}{4}\times(32.66)^2 = \frac{1}{4}\times 1067.11$$

$n = 266.77 \sim 267$

Example 4.15. In a random sample of 160 workers exposed to a certain amount of radiation 24 experienced some ill effects. Construct a 99% confidence interval for the corresponding true percentage.

(Supple. Nov. /Dec. 2004 Set 3)

Solution

Confidence interval for proportions p for large sample

$$\frac{x}{n} - z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}} < p < \frac{x}{n} + z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}$$

$z_{\alpha/2} = 2.58$ for $\alpha = 0.01$ level of significance

$1 - \alpha = 0.99$

$1 - 0.99 = \alpha$

$0.01 = \alpha$

$$\frac{24}{100} - 2.58\sqrt{\frac{\frac{24}{160}\left(1-\frac{24}{160}\right)}{160}} < p < \frac{24}{160} + 2.58\sqrt{\frac{\frac{24}{160}\left(1-\frac{24}{160}\right)}{160}}$$

$$0.15 - 2.58\sqrt{\frac{0.15(1-0.15)}{160}} < p < 0.15 + 2.58\sqrt{\frac{0.15(1-0.15)}{160}}$$

$0.15 - 2.58 \times 0.0282 < p < 0.15 + 2.58 \times 0.0282$

$0.15 - 0.0728 > p < 0.15 + 0.0728$

$0.07716 < p < 0.22283$ **Ans.**

Example 4.16. A random sample of 1200 apples was taken from a large consignment and found that 10% of them are bad. The supplier claims that only 2% are bad. Test his claim at 95% level.

(Reg. April / May 2005 Set)
(Supple. Feb. 2007 Set 3)
(Nov. 2006 Set 1)

Solution

1. Null hypothesis H_0: $p = 0.02$
2. Alternative hypothesis H_1: $p \neq 0.02$
3. Level of significance for 95 %, $\alpha = 0.05$ level of significance
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance.
5. Test of statistic

$$z = \frac{P-p}{\sqrt{pq/n}}$$

$$p = \frac{x}{n}$$ 10% of the 1200 apples are bad.

$$i.e. \frac{10}{100} \times \frac{1200}{1} = 120 \text{ Apples}$$

$$p = \frac{x}{n} = \frac{120}{1200} = 0.1$$

$$z = \frac{0.1 - 0.02}{\sqrt{\frac{(0.02)(1-0.02)}{1200}}} = \frac{0.08}{\sqrt{\frac{0.0196}{1200}}} = \frac{0.08}{0.0040} = 20$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted

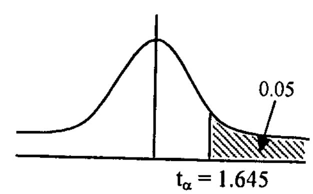
$-1.96 < 20 > 1.96$ null hypothesis is rejected.

Example 4.17. If a random sample of 120 pumps includes 47 required repairs within the first 5 years test the null hypothesis p = 0.30 against the alternative hypothesis p > 0.30 at 0.05 level of significance?

(Supple. Nov./Dec 2004 Set 2)

Solution

1. Null hypothesis H_0: p = .30
2. Alternative hypothesis H_1: p >0.30
3. Level of significance α = 0.05
4. Critical region: $z_\alpha = 1.645$ for α = 0.05 level of significance. Test is R.O.T.T
5. Test of statistic



$$z = \frac{P-p}{\sqrt{pq/n}}$$

$$p = \frac{47}{120}, p = 0.30$$

$$z = \frac{\frac{47}{120} - 0.30}{\sqrt{\frac{(0.30)(1-0.30)}{120}}} = \frac{0.39 - 0.30}{\sqrt{0.00175}} = \frac{0.09}{0.0418} = 2.15$$

6. Decision : $z < z_\alpha$ then null hypothesis is accept

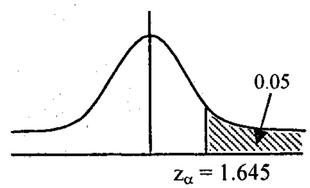
   2.15> 1.645 null hypothesis is rejected

Example 4.18. In a city 250 men out of 750 were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers?

(Reg. April / May 2005 Set 4)

Solution

1. Null hypothesis H_0: The number of smokers and non smokers are equal in the city $p = 0.5$
2. Alternative hypothesis H_1: $p > 0.5$
3. Level of significance $\alpha = .05$
4. Critical region: $z_\alpha = 1.645$ for $\alpha = 0.05$ level of significance. Test is R.O.T.T.



5. Test of statistic: $z = \dfrac{P-p}{\sqrt{pq/n}}$

$$P = \frac{x}{n} = \frac{250}{750} = 0.333$$

$$z = \frac{0.333-0.5}{\sqrt{(0.5)(1-0.5)/750}} = \frac{-0.1666}{\sqrt{0.00033}} = \frac{-0.1666}{0.01825}$$

$$= -9.128$$

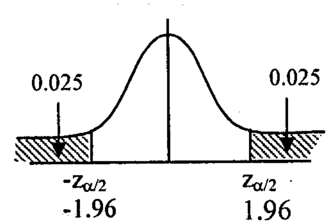
6. Decision: $z < z_\alpha$ then null hypothesis is accepted.

   -9.128 < 1.645 null hypothesis accepted

Example 4.19. A die is thrown 256 times an even digit turns up 150 times. Can we say that the die is unbiased? **(Supple. Nov. / Dec. 2005 Set 4)**

Solution

1. Null hypothesis H_0: die is unbiased
2. Alternative hypothesis H_1: die is biased
3. Level of significance $\alpha = 0.05$
4. Critical region: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$ level of significance. Test is T.T.T.



5. Test of statistic: $z = \dfrac{P-p}{\sqrt{pq/n}}$

$$p = \frac{x}{n} = \frac{150}{256} = 0.585$$

P = probability of getting even number (2 or 4 or 6)

$$= \frac{3}{6} = \frac{1}{2} = 0.5$$

$$z = \frac{0.585 - 0.5}{\sqrt{(0.5)(1-0.5)/256}} = \frac{0.085}{\sqrt{0.00097}} = \frac{0.085}{0.03125}$$

Z = 2.72

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accepted.

-1.96 < 2.72 > 1.96 null hypotheses is rejected

Exercise

(Test of Hypothesis one proportion)

1) A coin is tossed 1000 times and head turned up 200 times. Test whether the coin is biased?

[Hint: Null Hypothesis Ho: coin is unbiased

A.H H_1: coin is biased

L.O.S. $\alpha = 0.05$, $Z_{\alpha/2}=1.96$

Probability of getting head p=1/2, q=1/2

Test statistic= Z= (x-np)/√npq

= -300/15.811

= -18.974

-1.96>-18.974<1.96 , N.H is rejected, coin is biased]

2) A coin is tossed 960 times and head turned up 183 times. Is coin biased?

(Supple. JNTU 2004)

[Hint: Null Hypothesis Ho: coin is unbiased

A.H H_1: coin is biased

L.O.S. $\alpha = 0.05$, $Z_{\alpha/2}=1.96$

Probability of getting head p=1/2, q=1/2

Test statistic= Z= (x-np)/√npq

= -19.1

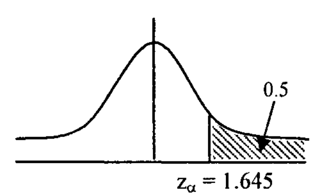
-1.96>-19.1<1.96 , N.H is rejected, coin is biased]

3) In Hyderabad city 3000 students out of 5000 were found to go to engineering colleges. Does this information support the conclusion that majority of the students in the city to go to engineering college.

[Hint: 1. N.H.H_0: The number of students who go to engineering colleges and other colleges are equal $p = 0.5$

2. A.H.H_1: $p > 0.5$

3. Level of significance $\alpha = .05$



4. Critical region: $z_\alpha = 1.645$, Test is R.O.T.T.

5. Test of statistic: $z = \dfrac{P-p}{\sqrt{pq/n}}$, $P = \dfrac{x}{n} = \dfrac{3000}{5000} = 0.6$

$$z = \frac{0.6-0.5}{\sqrt{(0.5)(1-0.5)/5000}} = 14.142$$

6. Decision: 14.142> 1.645 null hypothesis rejected.]

4) A random sample of 200 teachers, were taken from an engineering college for the teachers training programme, it is found that 10% of the teachers are not good in teaching. The principal of the college claims that only 2% teachers are not good in teaching. Test his claim at 95% level.

[Hint: 1. N.H.H_0: $p = 0.02$

2. A.H.H_1: $p \neq 0.02$

3. L.O.S. for 95 %, $\alpha = 0.05$

4. C.R.: $z_{\alpha/2}$ =1.96 for $\alpha = 0.05$ L.O.S.

5. Test of statistic

$$z = \frac{P-p}{\sqrt{pq/n}}$$

$P = \dfrac{x}{n}$ 10% of the 200 teachers *i.e.* $\dfrac{10}{100} \times \dfrac{200}{1} = 20$ Teachers

$$P = \frac{x}{n} = \frac{20}{200} = 0.1$$

$$z = \frac{0.1-0.02}{\sqrt{\dfrac{(0.02)(1-0.02)}{200}}} = 8.088$$

6. Decision: $-1.96 < 8.088 > 1.96$ null hypothesis is rejected.]

5) Among 900 people in Hyderabad city, 90 are found to take services of R.T.C buses to go to office every day. Construct 99% confidence interval for the true proportion.

[Hint: The confidence interval for P when n is large

$$\frac{x}{n} - z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}} < p < \frac{x}{n} + z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}, \; z_{\alpha/2} = 2.575 \text{ for 99 \% L.O.S.}$$

$$\frac{90}{900} - 2.575\sqrt{\frac{\frac{90}{900}\left(1-\frac{90}{900}\right)}{900}} < p < \frac{90}{900} + 2.575\sqrt{\frac{\frac{90}{900}\left(1-\frac{90}{900}\right)}{900}}$$

$0.07425 < p < 0.12575$ Ans.]

6) In a random
them are de
his product
[Hint: 1.
2.
3.
4.
5.
6.

1. A Sample
probability 0
(a) 0.87
Ans. (d)

2. In a random
quality of
correspondin
(a) (0.4, 0.7)
Ans. (b)

3. If p = 0.578,
true proporti
(a) 0.078
Ans. (b)

4. Among 100
99% confide
(a) (0.579, 0.
(c) (0.0742, (
Ans. (b)

6) In a random sample of 120 computer printers in an organization, It is found 10 of them are defective. The manufacturer claims that the percentage defective in his product doesn't exceed 6. Would you consider the claim justified?

[Hint: 1. N.H.H_0: $p = 0.06$

2. A.H.H_1: $p \neq 0.06$

3. L.O.S. for 95 %, $\alpha = 0.05$

4. C.R.: $-z_\alpha = -1.645$ for $\alpha = 0.05$ L.O.S.Test is L.O.T.T.

5. Test of statistic

$$z = \frac{P - p}{\sqrt{pq/n}},$$

$$P = \frac{x}{n} \quad \text{10\% of the 120 Printers } i.e. \frac{10}{100} \times \frac{120}{1} = 12 \text{ Printers}$$

$$P = \frac{x}{n} = \frac{12}{120} = 0.1$$

$$z = \frac{0.1 - 0.06}{\sqrt{\frac{(0.06)(1-0.06)}{120}}} = 1.851$$

6. Decision: $-1.96 < 1.851$ null hypothesis is accepted.]

OBJECTIVE TYPE QUESTIONS

1. A Sample of size 100 is taken whose standard deviation is 5.With the probability 0.95, what is the maximum error?

(a) 0.87 (b) 0.6 (c) 1 (d) 0.98

Ans. (d)

2. In a random sample of 400 items it is found that 231 are damage because of quality of production. Construct a 99% confidence interval for the corresponding true proportion.

(a) (0.4, 0.7) (b) (0.514, 0.642) (c) (0.5, 0.6) (d) (0.642, 0.751)

Ans. (b)

3. If p = 0.578, q=0.422, n = 400 find maximum error with 95% confidence for true proportion

(a) 0.078 (b) 0.048 (c) 0.058 (d) none of these

Ans. (b)

4. Among 100 people in a state 10 are found to be chapatti eaters. Construct 99% confidence interval for the true proportion.

(a) (0.579, 0.1258) (b) (0.0742, 0.1258)

(c) (0.0742, 0.1520) (d) none of these

Ans. (b)

5. A random sample of 100 bananas were taken and out of which 15 were found to be bad. What can we say with 95% confidence about the maximum error of proportion.

   (a) 0.03 (b) 0.05 (c) 0.07 (d) none of these

Ans. (c)

6. In a sample of 700 people in Andhra Pradesh 300 are rice eaters. What can you say about the maximum error with 99% confidence?

   (a) 0.0582 (b) 0.0482 (c) 0.0182 (d) none of these

Ans. (b)

7. If we can assert with 95% that the maximum error is 0.07 and p is 0.1. Find the size of the sample

   (a) 71 (b) 50 (c) 81 (d) none of these

Ans. (a)

8. If the maximum error with 99% probability is 0.25 and sample size n=500 then the variance of the population is

   (a) 2.166 (b) 3.205 (c) 4.694 (d) none of these

Ans. (c)

9. Among 100 faculty members in college 80 people use cell phone. With 95% confidence, the maximum error for true proportion is

   (a) 0.059 (b) 0.085 (c) 0.0784 (d) none of these

Ans. (c)

10. A random sample of 500 products, 100 are defective items. Standard error of Proportion with 99% is

   (a) 0.0329 (b) 0.0461 (c) 0.165 (d) 0.521

Ans. (b)

11. If p = 0.5 and the sample size is 70 then the maximum error with 99% confidence is

   (a) 0.567 (b) 0.1212 (c) 0.25 (d) 0.1541

Ans. (d)

12. If q = 0.3 and the sample size is 140 then the maximum error with 95% confidence is

   (a) 0.0459 (b) 0.0444 (c) 0.0559 (d) 0.0759

Ans. (d)

"

5.1 CONFIDENC

A $(1-\alpha)$ 100

Thus with the popul

One can assert witl

$z_{\alpha/2}\ \sigma/\sqrt{n}$ for the

The maximum error

With $(1-\alpha)$ probabi

5.2 CONFIDENCE

When $(n < 30)$,

h 15 were found
maximum error

) none of these

ers. What can

) none of these

p is 0.1. Find

) none of these

e size n=500

) none of these

none. With 95%

) none of these

andard error of

) 0.521

error with 99%

) 0.1541

error with 95%

) 0.0759

UNIT-5

SMALL SAMPLES

"Non-achievement doesn't mean, you are a failure,

It just means you haven't succeeded yet."

5.1 CONFIDENCE INTERVAL FOR μ

A $(1 - \alpha)$ 100% confidence internal for μ is given by

$$\bar{x} - z_{\alpha/2}\ \sigma/\sqrt{n} < \mu < \bar{x} + z_{\alpha/2}\ \sigma/\sqrt{n}$$

Thus with the population mean μ, sample mean $\bar{x}$ for a large sample $(n \geq 30)$. One can assert with probability $(1-\alpha)$ that the errors $|\bar{x} - \mu|$ will be less than $z_{\alpha/2}\ \sigma/\sqrt{n}$ for the small samples if σ is unknown or $(n < 30)$, σ is replaced by S. The maximum error estimate

$$E = t_{\alpha/2}\ S/\sqrt{n}$$

With $(1 - \alpha)$ probability, t-distribution is with $(n-1)$ degrees of freedom.

5.2 CONFIDENCE INTERVAL FOR μ FOR SMALL SAMPLES

When $(n < 30)$, assuming sampling from normal population

$$\bar{x} - t_{\alpha/2}\ S/\sqrt{n} < \mu < \bar{x} + t_{\alpha/2}\ S/\sqrt{n}$$

5.3 T – DISTRIBUTION

Or student t-distribution. Sampling distribution of mean (σ unknown)

If σ is unknown, for large samples (n ≥ 30) σ can be replaced by the sample standard deviation S.

$$S^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}$$

For small sample of size (n < 30) σ can be replaced by S. We make an assumption that the sample is drawn from a normal population.

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$$

Where

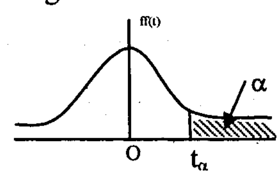
$\bar{X}$ → Mean of a random sample of size n drawn from a normal population

μ → Mean

σ^2 → Variance

t is a random variable having the t-distribution with ν = n -1 degrees of freedom

The t - distribution curve is symmetric about the mean 0. It is asymptotic on both sides of t -axis. Thus t - distribution curve is similar to normal curve. Critical values of t - distribution is denoted by t_α which is such that the area under the curve to the right of t_α equals to α.



5.4 TEST OF HYPOTHESIS (SMALL SAMPLES)

1. Single sample mean $\bar{x}$. With known sample variance S^2, σ unknown.

Working Procedure

For the small samples (n < 30), σ known, decision is based on the t- distribution with ν = n – 1 degrees of freedom.

1. Null hypothesis H_0: $\mu = \mu_0$
2. Alternative hypothesis H_1: $\mu \neq \mu_0$ or $\mu > \mu_0$ or $\mu < \mu_0$
3. Level of significance: α
4. Critical region

a. If $\mu \neq \mu_0$ test
from the t-distr

b. If $\mu > \mu_0$ test
distribution ta

c. If $\mu < \mu_0$ test
distribution tat

5. Test of statistic

Where $\bar{x}$
S is samp
n sample

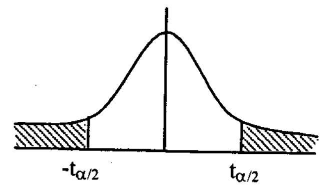
6. Decision

a. If $-t_{\alpha/2}$
Hypothesi

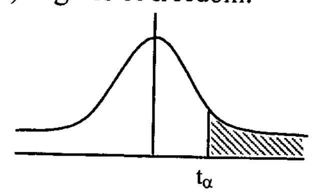
b. $t_\alpha > t$
Hypothesi

c. $t > -t_\alpha$
Hypothesi

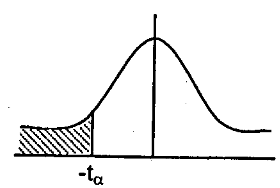
a. If $\mu \neq \mu_0$ test is two tail test, for given α, critical values are $-t_{\alpha/2}$ and $t_{\alpha/2}$ from the t-distribution table with (n – 1) degrees of freedom.



b. If $\mu > \mu_0$ test is right one tail test, for given α, critical value is t_α from the t-distribution table with (n-1) degrees of freedom.



c. If $\mu < \mu_0$ test is left one tail test, for given α, critical value is $-t_\alpha$ from the t-distribution table with (n-1) degrees of freedom.



5. Test of statistic

$$t = \frac{\bar{x} - \mu_0}{S / \sqrt{n}}$$

Where $\bar{x}$ is mean of the sample
S is sample standard deviation
n sample size

6. Decision

a. If $-t_{\alpha/2} < t < t_{\alpha/2}$
Hypothesis is accepted for two tail test otherwise rejected

b. $t_\alpha > t$
Hypothesis is accepted for right one tail test otherwise rejected

c. $t > -t_\alpha$
Hypothesis is accepted otherwise rejected. For left one tail test.

5.5 TEST OF HYPOTHESIS (SMALL SAMPLES)

Concerning difference between two means, with unknown σ_1^2 and σ_2^2 but equal.

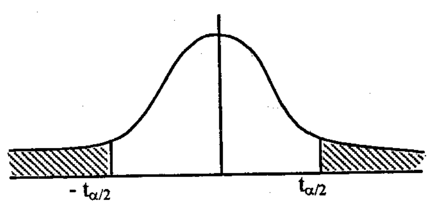
$(\sigma_1 = \sigma_2 = \sigma)$

Working Procedure

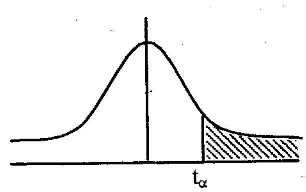
For the small samples $(n_1, n_2 < 30)$ and two sample are drawn from two normal population with population variances σ_1^2 and σ_2^2 unknown but equal $(\sigma_1 = \sigma_2 = \sigma)$. To test the hypothesis, concerning difference between two means, whether $\mu_1 - \mu_2 = 0$ or not then formulate.

1. Null hypothesis H_0: $\mu_1 - \mu_2 = 0$
2. Alternative hypothesis H_1: $\mu_1 - \mu_2 \neq 0$ or $\mu_1 - \mu_2 > 0$ or $\mu_1 - \mu_2 < 0$
3. Level of significance: α
4. Critical region

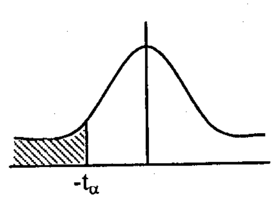
a. If $\mu_1 - \mu_2 \neq 0$ test is two tail test, for given, Critical values are $-t_{\alpha/2}$ and $t_{\alpha/2}$ from the t-distribution table with $n_1 + n_2 - 2$ degrees of freedom.



b. If $\mu_1 - \mu_2 > 0$ test is right one tail test, for given α, critical value is t_α from the t-distribution table with $n_1 + n_2 - 2$ degrees of freedom.



c. $\mu_1 - \mu_2 < 0$ test is left one tail test, for givenα, critical values is $-t_\alpha$ from the t-distribution table with $(n_1 + n_2 - 2)$ degrees of freedom.



5. Test of statistic
The statistic f
samples

$t = -$

With $(n_1 + n_2 -$
Where

$S_1^2 \rightarrow$ Fi
$S_2^2 \rightarrow$ Se
$\bar{x}_1 \rightarrow$ Fir
$\bar{x}_2 \rightarrow$ Sec
$n_1 \rightarrow$ Firs
$n_2 \rightarrow$ Sec

6. Decision:

a. If $-t_{\alpha/2} < t <$
Hypothesis is a

b. If $t_\alpha > t$
Hypothesis is a

c. If $t > -t_\alpha$
Hypothesis is a

5.6 TEST OF HYP

Two samples v
paired data by taking

1. Null hypo
2. Alternativ
3. Level of s
4. Critical re

a. If $\mu \neq 0$ test is
from the t-distri

5. Test of statistic

The statistic for test concerning difference between two means, for small samples

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}} \quad \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

With ($n_1 + n_2$ -2) degrees of freedom

Where

$S_1^2 \rightarrow$ First sample variance

$S_2^2 \rightarrow$ Second sample variance

$\bar{x}_1 \rightarrow$ First sample mean

$\bar{x}_2 \rightarrow$ Second sample mean

$n_1 \rightarrow$ First sample size

$n_2 \rightarrow$ Second sample size

6. Decision:

a. If $-t_{\alpha/2} < t < t_{\alpha/2}$

Hypothesis is accepted for two tail test otherwise rejected

b. If $t_\alpha > t$

Hypothesis is accepted for right one tail test otherwise rejected

c. If $t > -t_\alpha$

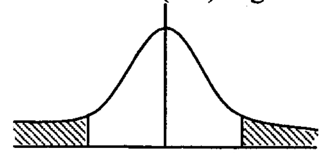
Hypothesis is accepted for left one tail test otherwise rejected.

5.6 TEST OF HYPOTHESIS (PAIRED SAMPLES)

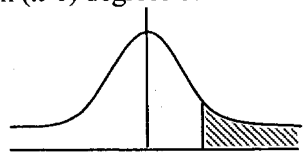
Two samples which are not independent, paired t-test can be applied for n paired data by taking the differences.

1. Null hypothesis H_0: $\mu = 0$
2. Alternative hypothesis H_1: $\mu \neq 0$ or $\mu > 0$ or $\mu < 0$
3. Level of significance: α
4. Critical region

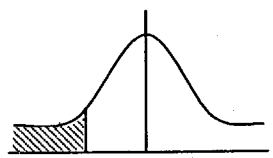
a. If $\mu \neq 0$ test is two tail test, for given α critical values are $-t_{\alpha/2}$ and $t_{\alpha/2}$ from the t-distribution table with (n-1) degrees of freedom



b. If $\mu > 0$ test is right one tail test, for given α critical values is t_α from the t-distribution table with (n-1) degrees of freedom



c. If $\mu < 0$ test is left one tail test, for given α critical values are $-t_\alpha$ from the t-distribution table with (n-1) degrees of freedom



5. Test of statistic

$$t = \frac{\bar{d} - \mu_d}{S_d / \sqrt{n}}$$

Where

$\bar{d}$ → mean of the differences

S_d → standard deviation of the differences

n → sample size

6. Decision

a. $-t_{\alpha/2} < t < t_{\alpha/2}$

Hypothesis is accepted for two tail test otherwise rejected

b. $t_\alpha > t$

Hypothesis is accepted for right one tail test otherwise rejected

c. $t > -t_\alpha$

Hypothesis is accepted for left one tail test otherwise rejected

5.7 F – DISTRIBUTION

Suppose we have a normal population with mean μ_1 and variance σ_1^2.and another normal population with mean μ_2 and variance σ_2^2.Two samples are drawn from the different population. First sample size is n_1 and sample variance is S_1^2. Second sample size is n_2 and sample variance is S_2^2.

The sampling distribu
random samples given

With $\nu_1 = n_1$-1 and ν_2 =
If the variances of the

F-distribution curve
quadrant. $F_2(\nu_1, \nu_2)$ i
with ν_1 and ν_2 d.o.f. s
the F – distribution cu
F_α is equal to α.

$F_{1-\alpha}(\nu_1, \nu_2$

$F_\alpha(\nu_1, \nu_2)$ Value can b
F - Distribution table. F

5.8 CHI-SQUARE D

The chi-square d
The distribution was
Probability Distributio
measure of goodness o

The chi-square dis

Definition-Let Z_1, Z_2,..
random variables with

$\aleph^2 = z_1^2$ -

Is called the chi-square
positive integer includ
density function

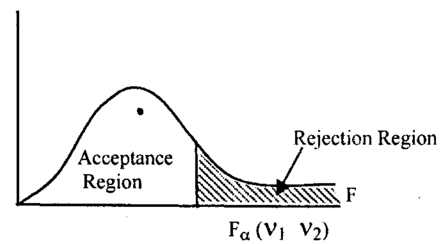
The sampling distribution of the ratio of the variances of the two independent random samples given by

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2}$$

With $\nu_1 = n_1-1$ and $\nu_2 = n_2-1$ degrees of freedom is known as F – distribution.
If the variances of the population are same i.e. $\sigma_1^2 = \sigma_2^2$ then

$$F = \frac{S_1^2}{S_2^2}$$

F-distribution curve lies in the first quadrant. $F_2(\nu_1, \nu_2)$ is the value of F with ν_1 and ν_2 d.o.f. s.t. the area under the F – distribution curve to the right of F_α is equal to α.



$$F_{1-\alpha}(\nu_1,\nu_2) = \frac{1}{F_\alpha(\nu_2,\nu_1)}$$

$F_\alpha(\nu_1,\nu_2)$ Value can be found by the F - Distribution table. For $\alpha = 0.05$ and $\alpha = 0.01$ level of significance.

5.8 CHI-SQUARE DISTRIBUTION ($\aleph^2$ - DISTRIBUTION)

The chi-square distribution, denoted by $\aleph^2$ (Here $\aleph$ is the Greek letter chi) The distribution was first derived by karl Pearson in 1900. It is a Continuous Probability Distribution of a continuous random variable X. It is mainly used as a measure of goodness of fit and to test the independence of variables.

The chi-square distribution is defined as

Definition-Let $Z_1, Z_2, \ldots\ldots\ldots\ldots, Z_K$ be k independent ,Standard, normally distributed random variables with mean $\mu = 0$ and variance $\sigma^2 = 1$.Then the random variable

$$\aleph^2 = z_1^2 + z_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots z_k^2$$

Is called the chi-square distribution with k degrees of freedom. Which can be any positive integer including 1, is denoted by dof.$\aleph^2$ - Distribution has the probability density function

$$f(x) = \begin{cases} \dfrac{x^{(k-2)/2}e^{-x/2}}{2^{k/2}\Gamma(\frac{k}{2})} & x > 0 \\ 0 & else\ where \end{cases}$$

Here $\Gamma(.)$ is the gamma function defined as follows

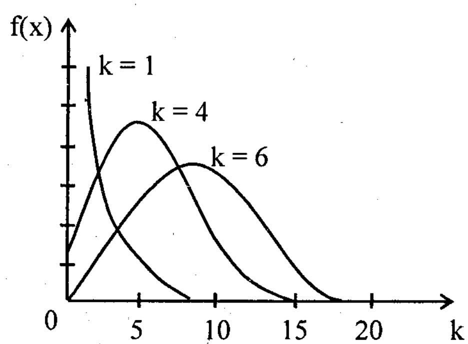
$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx$$

The mean and variance of the $\aleph^2$ - Distribution

$\mu = k$ and variance $\sigma^2 = 2k$

There is a $\aleph^2$ - Distribution for each k. Several chi-square distributions are shown in figure. The chi-square random variable is non-negative; the distribution is not symmetric and is skewed to the right. As k increases, the distribution becomes more symmetric. For large k the distribution is close to the normal distribution.
In other words

As $k \to \infty$, the limiting form of the chi-square distribution is the normal distribution.



Chi-square distribution for k degrees of freedom

5.9 GOODNESS OF FIT ($\aleph^2$ - TEST)

Suppose we have a collection of data of observed frequencies which is obtained from some experiment. We want to decide by some test the data of observed frequencies are fit or not in some specific distribution. Then the null hypothesisH_0: The data are fit in given distribution if observed data is denoted by O and expected data denoted by E, the chi-square statistic is

$$\aleph^2 = \sum \frac{(Observed - Expected)^2}{Expected} = \sum \frac{(O-E)^2}{E}$$

This test is known as $\aleph^2$ test of goodness of fit.
Suppose a collection of data, given by a frequency distribution with n categories then degrees of freedom=n-1

If distribution is Binomial distribution dof = n-1
If distribution is Poison distribution dof = n-2

If distribution is No
If $\aleph^2$ value is too lar
that the fit is poor
significance level α.(
From $\aleph^2$-distributi
significance level α.
graph. If $\aleph^2$ value ex
is rejected at the α sig

5.10 TEST OF HYI
Suppose we hav
P_3.......P_k, with the
against the alternative

1. Null Hypot
2. Alternative
3. Level of sig
4. Critical Re
   hypothesis
   null hypoth
5. Test of stati

Where e_{ij} ar

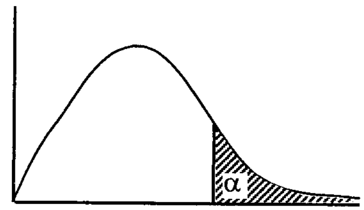
$x \to$ total nu
$n \to$ total nu

6. Decision: $\aleph \leq \aleph^2_{...}$
   accepted otherwis

If distribution is Normal distribution dof = n-3

If $\aleph^2$ value is too large, if $\aleph^2$ exceeds from given critical value c, then we can say that the fit is poor and we reject H_0.The critical value c can be obtained by a significance level α.Generally α=0.10, α=0.05 or α=0.005 can be used.

From $\aleph^2$-distribution table, critical values can be obtained for different significance level α. The significance level α represents the shaded area in the graph. If $\aleph^2$ value exceeds the critical value c, then we say that null hypothesis H_0 is rejected at the α significance level.



Critical value c

5.10 TEST OF HYPOTHESIS CONCERNING SEVERAL PROPORTIONS

Suppose we have to test k binomial population, with the parameters P_1, P_2, P_3.......P_k, with the null hypothesis that k populations proportions are all equal against the alternative hypothesis that proportions are not all equal.

1. Null Hypothesis H_0: $P_1 - P_2 = ... = P_k$
2. Alternative Hypothesis H_1: P_1, P_2, P_3.......P_k is not all equal
3. Level of significance: α
4. Critical Region: $\aleph \leq \aleph_\alpha^2$ with (k-1) degrees of freedom the null hypothesis is accepted if $\aleph^2 > \aleph_\alpha^2$ with (k-1) degrees of freedom then null hypothesis is rejected.
5. Test of statistic

$$\aleph^2 = \sum_{i=1}^{2} \sum_{j=1}^{k} \frac{(O_{ij} - e_{ij})^2}{e_{ij}}$$

Where e_{ij} are given by

$$e_{ij} = \frac{n_j.x}{n},\ e_{2j} = \frac{n_j(n-x)}{n}$$

x → total number of success for all samples

n → total number of trials for all samples

6. Decision: $\aleph \leq \aleph_\alpha^2$ with (k-1) degrees of freedom then null hypothesis is accepted otherwise rejected

SOLVED EXAMPLES

Example 5.1. Ten bearings made by a certain process have a mean diameter of 0.5060 c.m. with a standard deviation of 0.0040 cm. assuming that the data may be looked up on as a random sample from a normal distribution, construct a 95%. Confidence interval for the actual average diameter of the bearings?

(Supple. Nov. /Dec. 2004 Set 3)

(Reg. April / May 2005 Set 3)

Solution: Given that

Sample size n = 10

Mean $\bar{x} = 0.5060$ c.m.

Standard deviation S = 0.0040 c.m.

$z_{\alpha/2} = 1.96$ for confidence level $\alpha = 95\%$

Confidence interval =?

$$\bar{x} - t_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{S}{\sqrt{n}}$$

$$1 - \alpha = 0.95$$

$$1 - 0.95 = \alpha$$

$$0.05 = \alpha$$

$$t_{\alpha/2} = t_{0.05/2} = t_{0.025} = 2.262 \text{ For } \nu = 10 - 1 = 9$$

$$0.5060 - 2.262 \times \frac{0.0040}{\sqrt{10}} < \mu < 0.5060 + 2.262 \times \frac{0.0040}{\sqrt{10}}$$

$$0.5060 - \frac{0.009048}{3.1622} < \mu < 0.5060 + \frac{0.009048}{3.1622}$$

$$0.5060 - 0.00286 < \mu < 0.5060 + 0.00286$$

$$0.5031 < \mu < 0.5088$$ **Ans.**

Example 5.2. A sample of 10 cam shafts intended for use in gasoline engines has an average eccentricity of 1.02 and a standard deviation of 0.044 inch. Assuming the data may be treated a random sample from a normal population; determine a 95% confidence interval for the actual mean eccentricity of the cam shaft.

(Supple. Nov. /Dec. 2006 Set 4)

Solution : Given that

Sample size n = 10

Sample mean $\bar{x} = 1.02$

Standard deviation S = 0.044

$$1 - \alpha = 0.95$$

$$1 - 0.95 = \alpha$$

$$0.05 = \alpha$$

$$t_{\alpha/2} = t_{0.05/2} = t_{.025} = 2.262 \text{ For } \nu = 10 - 1 = 9 \text{ d.o.f.}$$

Confidence interval =?

$$\bar{x} - t_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{S}{\sqrt{n}}$$

$$1.02 - 2.262 \times \frac{0.044}{\sqrt{10}} < \mu < 1.02 + 2.262 \times \frac{0.044}{\sqrt{10}}$$

$$1.02 - \frac{0.09952}{3.1622} < \mu < 1.02 + \frac{0.09952}{3.1622}$$

$$1.02 - 0.0314 < \mu < 1.02 + 0.03147$$

$$0.98852 < \mu < 1.0514$$

Confidence interval is (0.98852, 1.0514) **Ans.**

Example 5.3. In an air pollution study, the following amounts of suspended benzene soluble organic matter (in micrograms per cubic meter) were obtained at an experiment station for eight different samples of air. 2.2, 1.8, 3.1, 2.0, 2.4, 2.0, 2.1 & 1.2 .Construct a 0.95 confidence interval for the corresponding true mean.

(Reg. April / May 2004)

Solution

$$\bar{x} = \frac{2.2 + 1.8 + 3.1 + 2.0 + 2.4 + 2.0 + 2.1 + 1.2}{8}$$

$$= \frac{16.8}{8}$$

$$\bar{x} = 2.1$$

$$S^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

$$= \frac{(2.2-2.1)^2 + (1.8-2.1)^2 + (3.1-2.1)^2 + (2.0-2.1)^2 + (2.4-2.1)^2 + (2.0-2.1)^2 + (2.1-2.1)^2 + \left(1.2-2.1)^2\right)}{8-1}$$

$$= \frac{(0.1)^2 + (-0.3)^2 + (1)^2 + (-0.1)^2 + (0.3)^2 + (-0.1)^2 + (0)^2 + (-0.9)^2}{7}$$

$$= \frac{0.01 + 0.09 + 1 + 0.01 + 0.09 + 0.01 + 0.81}{7}$$

$$= \frac{2.02}{7}$$

$$S^2 = 0.288$$
$$S = 0.53718$$

Confidence interval is

$$\bar{x} - t_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{S}{\sqrt{n}}$$

$t_{0.05/2} = t_{0.025} = 2.365$ From the table with n-1 = 8- 1 = 7 d.o.f.

$$2.1 - 2.365 \times \frac{0.53718}{\sqrt{8}} < \mu < 2.1 + 2.365 \times \frac{0.53718}{\sqrt{8}}$$

$$2.1 - \frac{1.2704}{2.8284} < \mu < 2.1 + \frac{1.2704}{2.8284}$$

$$2.1 - 0.44915 < \mu < 2.1 + 0.44915$$

$$1.6508 < \mu < 2.5491$$

Example 5.4. A random sample of 6 steel beams has a mean compressive strength of 58, 392 P.S.I (pounds per square inch) with a standard deviation of 648 P.S.I. Use this information at the level of significance $\alpha = 0.05$ to test whether the true coverage compressing strength of steel from which this sample came is 58,000 P.S.I assume normality?

(Reg. April / May 2005 Set 2)

Solution

1. Null hypothesis H_0: $\mu = 58000$
2. Alternative hypothesis H_1: $\mu \neq 58000$
3. Level of significance: $\alpha = 0.05$
4. Critical region: $t_{\alpha/2} = t_{0.05/2} = t_{0.025} = 2.571$ for n-1 = 6 – 1 = 5 d.o.f.
5. Test of statistic:

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$$

Sample mean $\bar{x} = 58392$
Population mean $\mu = 58000$
Sample standard deviation S = 648
Sample size n = 6

$$t = \frac{58392 - 58000}{648/\sqrt{6}} = \frac{392}{648/2.449}$$

6. Decision:
- 2

Example 5.5. Two
following values

Sample I	
Sample II	

Is the difference betw

Solution

1. Null hypo
2. Alternativ
3. Level of si
4. Critical re

With $n_1 + n_2 -$

5. Test statist

$t = -$

With $n_1 + n_2 -$

Mean of the sam

Mean of the sam

$$= \frac{392}{264.59} = 1.481$$

6. Decision: $-t_{\alpha/2} < t < t_{\alpha/2}$ then null hypothesis is accepted
   $-2.571 < 1.481 < 2.571$ null hypothesis is accepted.

Example 5.5. Two independent samples of 8 and 7 items respectively had the following values **(Reg. April / May 2005)**

(Supple. Feb. 2007)

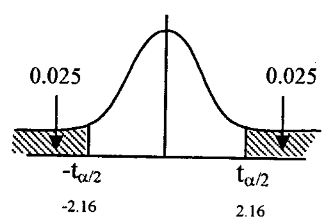
Sample I	11	11	13	11	15	9	12	14
Sample II	9	11	10	13	9	8	10	-

Is the difference between the means of samples significant?

Solution

1. Null hypothesis: $\mu_1 = \mu_2$
2. Alternative hypothesis: $\mu_1 \neq \mu_2$
3. Level of significance: $\alpha = 0.05$
4. Critical region: Test is two tailed test $t_{\alpha/2} = t_{.05/2} = t_{.025}$

With $n_1 + n_2 - 2 = 8 + 7 - 2 = 13$ degrees of freedom $t_{.025} = 2.16$



5. Test statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

With $n_1 + n_2 - 2$ d.o.f.

Mean of the sample one $\bar{x}_1 = \frac{\sum x_i}{n_1} = \frac{11+11+13+11+15+9+12+14}{8}$

$$= \frac{96}{8} = 12$$

Mean of the sample two $\bar{x}_2 = \frac{\sum x_i}{n_2} = \frac{9+11+10+13+9+8+10}{7}$

$$= \frac{70}{7} = 10$$

$$S_1^2 = \frac{\sum (x_i - \bar{x}_1)^2}{n_1 - 1}$$

$$= \frac{(11-12)^2 + (11-12)^2 + (13-12)^2 + (11-11)^2 + (15-12)^2 + (9-12)^2 + (12-12)^2 + (14-12)^2}{8-1}$$

$$= \frac{(-1)^2 + (-1)^2 + (1)^2 + (-1)^2 + (3)^2 + (-3)^2 + (0)^2 + (2)^2}{7}$$

$$= \frac{1+1+1+1+9+9+4}{7} = \frac{26}{7} = 3.714$$

$$S_2^2 = \frac{\sum (x_i - \bar{x}_1)^2}{n_2 - 1}$$

$$= \frac{(9-10)^2 + (11-10)^2 + (10-10)^2 + (13-10)^2 + (9-10)^2 + (8-10)^2 + (10-10)^2}{7-1}$$

$$= \frac{(-1)^2 + (1)^2 + (0)^2 + (3)^2 + (-1)^2 + (-2)^2 + (0)^2}{6}$$

$$= \frac{1+1+0+9+1+4+0}{6} = \frac{16}{6} = 2.666$$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\sqrt{(n_1-1)S_1^2 + (n_2-1)S_2^2}}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

$$= \frac{(12-10) - 0}{\sqrt{(8-1)(3.714) + (7-1)(2.666)}} \sqrt{\frac{8 \times 7(8+7-2)}{8+7}}$$

$$= \frac{2}{\sqrt{7 \times 3.714 + 6(2.666)}} \times \sqrt{\frac{56 \times 13}{15}} = \frac{2}{\sqrt{25.998 + 15.996}} \times \sqrt{48.533}$$

$$= \frac{2}{\sqrt{41.994}}(6.9665) = \frac{13.933}{6.4802} = 2.150$$

6. Decision

$-t_{\alpha/2} < t < t_{\alpha/2}$ Then null

Hypothesis is accepted
$-2.16 < 2.150 < 2.16$
i.e. null hypothesis is accepted
The difference is not significant

Example 5.6. Measurements of the fat content of two kinds of ice creams brand A and brand B yielded the following sample data

(Reg. April / May 2005 Set 2)

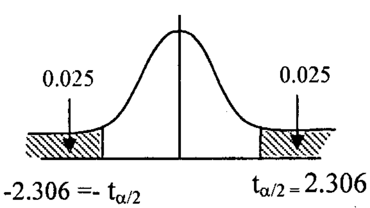
Brand A	13.5	14.0	13.6	12.9	13.0
Brand B	12.9	13.0	12.4	13.5	12.7

Test the significant between the means at 0.05 level

Solution

1. Null hypothesis: $\mu_1 = \mu_2$
2. Alternative hypothesis: $\mu_1 \neq \mu_2$
3. Level of significance: $\alpha = 0.05$
4. Critical region: $t_{\alpha/2} = t_{.05/2} = t_{.025} = 2.306$

With $n_1 + n_2 - 2 = 5 + 5 - 2 = 8$ degrees of freedom



5. Test statistic:

$$t = \frac{(\bar{x}_A - \bar{x}_B) - (\mu_1 - \mu_2)}{\sqrt{(n_A - 1)S_A^2 + (n_B - 1)S_B^2}} \sqrt{\frac{n_A n_B (n_A + n_B - 2)}{n_A + n_B}}$$

With $n_1 + n_2 - 2$ d.o.f.

Mean of the brand A

$$\bar{x}_A = \frac{13.5 + 14.0 + 13.6 + 12.9 + 13.0}{5} = \frac{67}{5} = 13.4$$

Mean of the brand B

$$\bar{x}_B = \frac{12.9 + 13.0 + 12.4 + 13.5 + 12.7}{5} = \frac{64.5}{5} = 12.9$$

$$S_A^2 = \frac{\sum (x_i - \bar{x}_A)^2}{n_1 - 1}$$

$$= \frac{(13.5-13.4)^2 + (14-13.4)^2 + (13.6-13.4)^2 + (12.9-13.4)^2 + (13-13.4)^2}{5-1}$$

$$= \frac{(0.1)^2 + (0.6)^2 + (0.2)^2 + (-0.5)^2 + (-0.4)^2}{4}$$

$$= \frac{0.01 + 0.36 + 0.04 + 0.25 + 0.16}{4}$$

$$= \frac{0.82}{4}$$

$$= 0.205$$

$$S_B^2 = \frac{\sum (x_i - \bar{x}_B)^2}{n_2 - 1}$$

$$= \frac{(12.9-12.9)^2 + (13-12.9)^2 + (12.4-12.9)^2 + (13.5-12.9)^2 + (12.7-12.9)^2}{5-1}$$

$$= \frac{(0)^2 + (0.1)^2 + (-0.5)^2 + (0.6)^2 + (-0.2)^2}{4}$$

$$= \frac{0 + 0.01 + 0.25 + 0.36 + 0.04}{4}$$

$$= \frac{0.66}{4} = 0.165$$

$$t = \frac{(\bar{x}_A - \bar{x}_B) - (\mu_1 - \mu_2)}{\sqrt{(n_A - 1)S_A^2 + (n_B - 1)S_B^2}} \sqrt{\frac{n_A n_B (n_A + n_B - 2)}{n_A + n_B}}$$

$$= \frac{(13.4 - 12.9) - 0}{\sqrt{(5-1)(0.205) + (5-1)(0.165)}} \sqrt{\frac{5 \times 5 (5+5-2)}{5+5}}$$

$$= \frac{(0.5)}{\sqrt{0.82 + 0.66}} \frac{\sqrt{25 \times 8}}{10}$$

$$= \frac{0.5}{1.2165}(4.472)$$

$$t = 1.8381$$

6. Decision

$-t_{\alpha/2} < t < t_{\alpha/2}$ then null Hypothesis is accepted

- 2.306 < 1.8381 < 2.306 null hypothesis is accepted

Example 5.7. Mea
was found that 8 s
standard deviation
denier of 7.43 with
test the hypothesis
significance?

Solution

1. Null hyp
2. Alternati
3. Level of
4. Critical r

With $n_1 + n_2$ -

5. Test statis

$t = \frac{(\bar{x}_1 - \bar{x}}{\sqrt{\sqrt{(n_1 - 1}}}$

First sample m
First sample st
Second sample
Second sample

$t = \frac{(9.6}{\sqrt{(8-1)(1.}}$

$= \frac{}{\sqrt{7 \times 3.}}$

$= \frac{}{\sqrt{22.93}}$

$= \frac{(0.74)(8}{6.53}$

$= 0.9555$

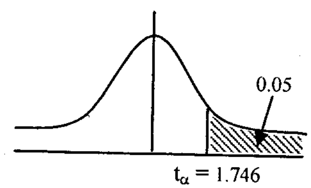
6. Decision: t
0.9555 < 1

Example 5.7. Measuring specimens of nylon yarn taken from two machines. It was found that 8 specimens from 1st machine had a mean denier of 9.67 with a standard deviation of 1.81 while 10 specimens from a 2nd machine had a mean denier of 7.43 with a standard deviation 1.48. Assuming the population are normal test the hypothesis $H_0 : \mu_1 - \mu_2 = 1.5$ against $H_1 : \mu_1 - \mu_2 > 1.5$ at 0.05 level of significance? **(Reg. April / May 2005 Set 2)**

Solution

1. Null hypothesis H_0: $\mu_1 - \mu_2 = 1.5$
2. Alternative hypothesis H_1: $\mu_1 - \mu_2 > 1.5$
3. Level of significance: $\alpha = 0.05$
4. Critical region: $t_\alpha = 1.746$

With $n_1 + n_2 - 2 = 8 + 10 - 2 = 16$ degrees of freedom. Test is R.O.T.T.



5. Test statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\sqrt{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

First sample mean $\bar{x}_1 = 9.67$

First sample standard deviation $S_1 = 1.81$

Second sample mean $\bar{x}_2 = 7.43$

Second sample standard deviation $S_2 = 1.48$

$$t = \frac{(9.67 - 7.43) - (1.5)}{\sqrt{(8-1)(1.81)^2 + (10-1)(1.48)^2}} \sqrt{\frac{8 \times 10(8 + 10.2)}{8 + 10}}$$

$$= \frac{2.24 - 1.5}{\sqrt{7 \times 3.2761 + 9 \times 2.1904}} \times \sqrt{\frac{80 \times 16}{18}}$$

$$= \frac{0.74}{\sqrt{22.9327 + 19.7136}} \times \sqrt{71.111}$$

$$= \frac{(0.74)(8.4327)}{6.53041}$$

$= 0.9555$

6. Decision: $t < t_\alpha$ then null hypothesis is accepted
   $0.9555 < 1.746$ null hypotheses is accepted

Example 5.8. Independent random samples of the heights of adult males living in two countries yielded the following results: n = 12, ~x = 65.7 inches, s_x = 4 inches and m = 15, ~y = 68.2 inches, s_y = 3 inches. Find an approximate 98% confidence interval for the difference $\mu_x - \mu_y$ of the means of the populations of heights assume that $\sigma_x^2 - \sigma_y^2$.

(Reg. April / May 2004)

Solution

Given that

$n = 12,\ \bar{x} = 65.7,\ s_x = 4$

$m = 15,\ \bar{y} = 68.2,\ s_y = 3$

Pooling Variance σ^2 is given by

$$\sigma^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Where s_1^2 *and* s_2^2 are the variance of two samples of size n_1 and n_2 respectively.

$$S^2 = \frac{(12-1)16 + (15-1)9}{12 + 15 + -2} = \frac{11 \times 16 + 14 \times 9}{25} = \frac{176 + 126}{25}$$

$$= \frac{302}{25} = 12.08$$

S = 3.475,

$1-\alpha = 0.98$

$1-0.98 = \alpha$

$0.02 = \alpha$

$t_{\alpha/2} = t_{0.02/2} = t_{0.01} = 2.485$ For $n_1 + n_2 - 2 = 12 + 15 - 2 = 25$ d.o.f.

$(1-\alpha)$ 100% confidence interval for the difference $\mu_n - \mu_y$

$$(\bar{x} - \bar{y}) - t_{\alpha/2}.S\sqrt{\frac{1}{n} + \frac{1}{m}} < \mu_x - \mu_y < (\bar{x} - \bar{y}) + t_{\alpha/2}.S\sqrt{\frac{1}{n} + \frac{1}{m}}$$

$$(65.7 - 68.2) - 2.485 \times 3.475\sqrt{\frac{1}{12} + \frac{1}{15}} < \mu_x - \mu_y$$

$$< (65.7 - 68.2) + 2.485 \times 3.475\sqrt{\frac{1}{12} + \frac{1}{15}}$$

$$-2.5 - 8.6353 \times \sqrt{0.083 + 0.066} < \mu_x - \mu_y < -2.5 + 8.6353\sqrt{0.083 + 0.061}$$

$2.5 - 3.3407 < \mu_x - \mu_y < -2.5 + 3.3407$

$-5.8407 < \mu_x - \mu_y < 0.8407$

Example 5.9. To ex
the wives. An invest
which measures the

Husbands	1
Wives	1

Test the hypothesis v

Solution

1. Null hypo
2. Alternativ
3. Level of si
4. Critical reg

With n_1 +

5. Test statisti

$t =$

With $n_1 + n_2 - 2$
Mean of the sam

$\bar{x}_1 = \frac{117 + 105}{}$

$= \frac{1030}{10} = 1$

Mean of the sam

$\bar{x}_2 = \frac{106 + 98}{}$

$= \frac{958}{10} = 95$

Example 5.9. To examine the hypothesis that the husbands are more intelligent than the wives. An investigator took a sample of 10 couples and administered them a test which measures the IQ as follows

(Supple. Nov./Dec. 2005 Set 4)

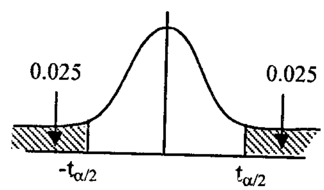
Husbands	117	105	97	105	123	109	86	78	103	107
Wives	106	98	87	104	116	95	90	69	108	85

Test the hypothesis with a reasonable test at the level of significance of 0.05?

Solution

1. Null hypothesis: $\mu_1 = \mu_2$
2. Alternative hypothesis: $\mu_1 \neq \mu_2$
3. Level of significance: $\alpha = 0.05$
4. Critical region: Test is two tailed test $t_{\alpha/2} = t_{.05/2} = t_{.025} = 2.101$

   With $n_1 + n_2 - 2 = 10 + 10 - 2 = 18$ degrees of freedom



5. Test statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

With $n_1 + n_2 - 2$ d.o.f.

Mean of the sample one

$$\bar{x}_1 = \frac{117 + 105 + 97 + 105 + 123 + 109 + 86 + 78 + 103 + 107}{10}$$

$$= \frac{1030}{10} = 103$$

Mean of the sample two

$$\bar{x}_2 = \frac{106 + 98 + 87 + 104 + 116 + 95 + 90 + 69 + 108 + 85}{10}$$

$$= \frac{958}{10} = 95.8$$

$$S_1^2 = \frac{\sum (x_i - \bar{x}_1)^2}{n_1 - 1}$$

$$= \frac{(117-103)^2 + (105-103)^2 + (97-103)^2 + (105-103)^2 + (123-103)^2 + (109-103)^2 + (86-103)^2 + (78-103)^2 + (78-103)^2 + (103-103)^2 + (107-103)^2}{10-1}$$

$$= \frac{(14)^2 + (2)^2 + (-6)^2 + (2)^2 + (20)^2 + (6)^2 + (-17)^2 + (-25)^2 + (0)^2 + (4)^2}{9}$$

$$= \frac{1606}{9}$$

$$S_1^2 = 178.4$$

$$S_1 = 13.35$$

$$S_2^2 = \frac{\sum (x_i - \bar{x}_2)^2}{n_2 - 1}$$

$$= \frac{(106-95.8)^2 + (98-95.8)^2 + (87-95.8)^2 + (104-95.8)^2 + (116-95.8)^2 + (95-95.8)^2 + (90-95.8)^2 + (69-95.8)^2 + (108-95.8)^2 + (85-95.8)^2}{10-1}$$

$$= \frac{(10.2)^2 + (2.2)^2 + (-8.8)^2 + (8.2)^2 + (20.2)^2 + (-0.8)^2 + (-5.8)^2 + (-26.8)^2 + (12.2)^2 + (-10.8)^2}{9}$$

$$S_2^2 = \frac{104.04 + 4.84 + 77.44 + 67.24 + 408.04 + 0.64 + 33.64 + 718.24 + 148.84 + 116.60}{9}$$

$$= \frac{1679.6}{9}$$

$$S_2^2 = 186.62$$

$S_2 = 13.66$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1}{\sqrt{(n_1 - 1)S_1^2 + (n}}$$

$$= \frac{(103}{\sqrt{(10-1)(178.}}$$

$$= \frac{}{\sqrt{9 \times 178}}$$

$$= \frac{7}{\sqrt{1605.6}}$$

$$= \frac{7.2 \times 9.4}{57.316}$$

t = 1.191

6. Decision: –

-2.101 < 1.1

Example 5.10. To co mounted on a car and the costs of repairs.

Guar
Guar

Use the 0.01 leve sample means is s

Solution

1. Null hypoth
2. Alternative l
3. Level of sig
4. Critical regi

With n_1+n_2-

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

$$= \frac{(103 - 95.8) - 0}{\sqrt{(10-1)(178.4) + (10-1)(186.62)}} \times \sqrt{\frac{10 \times 10(10 + 10 - 2))}{10 + 10}}$$

$$= \frac{7.2}{\sqrt{9 \times 178.4 + 9 \times 186.62}} \times (9.486)$$

$$= \frac{7.2}{\sqrt{1605.6 + 1679.78}} \times (9.486)$$

$$= \frac{7.2 \times 9.486}{57.316}$$

$t = 1.191$

6. Decision: $-t_{\alpha/2} < z < t_{\alpha/2}$ then null hypothesis is accept
   -2.101 < 1.191 < 2.101 null hypothesis is accepted

Example 5.10. To compare two kinds of bumper guards, 6 od each kind were mounted on a car and then the car has ran into a concrete wall. The fallowing are the costs of repairs.

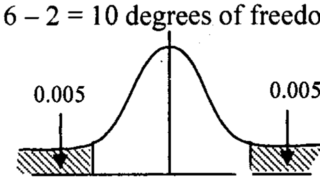
(Supple. Nov. / Dec. 2005 Set 4)

Guard 1	107	148	123	165	102	119
Guard 2	134	115	112	151	133	129

Use the 0.01 level of significance to test whether the difference between two sample means is significant?

Solution

1. Null hypothesis: $\mu_1 - \mu_2 = 0$
2. Alternative hypothesis: $\mu_1 - \mu_2 \neq 0$
3. Level of significance: $\alpha = 0.01$
4. Critical region: Test is two tailed test $t_{\alpha/2} = t_{.01/2} = t_{.005} = 3.169$
   With $n_1+n_2-2 = 6 + 6 - 2 = 10$ degrees of freedom. Test is two tailed test



5. Test of statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

With $(n_1 + n_2 - 2)$ d.o.f.

Mean of first sample of guard one

$$\bar{x}_1 = \frac{107 + 148 + 123 + 165 + 102 + 119}{6}$$

$$= \frac{764}{6} = 127.3$$

Mean of second sample of guard two

$$\bar{x}_2 = \frac{134 + 115 + 112 + 151 + 133 + 129}{6}$$

$$= \frac{774}{6} = 129$$

$$S_1^2 = \frac{\sum (x_i - \bar{x}_1)^2}{n_1 - 1}$$

$$= \frac{(107 - 127.3)^2 + (148 - 127.3)^2 + (123 - 127.3)^2 + (165 - 127.3)^2 + (102 - 127.3)^2 + (119 - 127.3)^2}{6 - 1}$$

$$= \frac{(-20.3)^2 + (20.7)^2 + (-4.3)^2 + (37.7)^2 + (-25.3)^2 + (-8.3)^2}{5}$$

$$= \frac{412.09 + 428.49 + 18.49 + 1421.29 + (640.00) + 68.89}{5}$$

$$= \frac{2989.34}{5}$$

$$S_1^2 = 597.868$$

$$S_1 = 24.451$$

$$S_2^2 = \frac{\sum (x_i - \bar{x}_2)^2}{n_2 - 1}$$

$$= \frac{(134-129)^2 + (115-129)^2 + (112-129)^2 + (151-129)^2 + (133-129)^2 + (129-129)^2}{6-1}$$

$$= \frac{(52)^2 + (-14)^2 + (17)^2 + (22)^2 + (4)^2 + (0)^2}{5}$$

$$= \frac{25 + 196 + 289 + 484 + 16 + 0}{5}$$

$$S_2^2 = \frac{1010}{5} = 202$$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

$$= \frac{(127.3 - 129)}{\sqrt{(6-1)(597.86) + (6-1)202}} \sqrt{\frac{6 \times 6(6+6-2)}{6+6}}$$

$$= \frac{-1.7}{\sqrt{2989.3 + 1010}} \times 5.477$$

$$= -\frac{9.3109}{\sqrt{3999.3}}$$

$$= -\frac{9.3109}{63.2400}$$

$$= -0.147$$

6. Decision: $-z_{\alpha/2} < z < z_{\alpha/2}$ then null hypothesis is accept

   $-3.169 < -0.147 < 3.169$ null hypothesis is accepted

Example 5.11. Ten soldiers participated in a shooting competition in the first week. After intensive training they participated in the competition in the second week their scores before and after training given as follows.

(Reg. April / May 2005 Set 2)

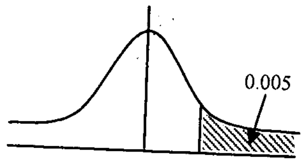
Scores before	67	24	57	55	63	54	56	68	33
Scores after	70	38	58	58	56	67	68	75	42

Do the data indicate that the soldiers have been benefited by the training.

Solution

1. Null hypothesis: $\mu = 0$ i.e. training is not useful
2. Alternative hypothesis: $\mu > 0$ i.e. training

3. Level of significance: $\alpha = 0.01$
4. Critical region: Test is two tailed test $t_{\alpha/2} = t_{0.01} = 2.821$
   With = 10 - 1 = 9 degrees of freedom. Test is R.O.T.T.



5. Test of statistic:

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

Differences di's are

-3, -14, -1, -3, 7, -13, -12, -7, -9, 5

$\bar{d}$ = mean of the differences of sample data = $\frac{-50}{10} = -5$

$$S_d^2 = \frac{[-3-(-5)]^2 + [-14-(-5)]^2 + [-1-(-5)]^2 + [-3-(-5)]^2 + [7-(-5)]^2 + [-13-(-5)]^2 + [-12-(-5)]^2 + [-7-(-5)]^2 + [-9-(-5)]^2 + [5-(-5)]^2}{10-1}$$

$$= \frac{(-3+5)^2 + (14+5)^2 + (-1+5)^2 + (-3+5)^2 + (7+5)^2 + (-13+5)^2 + (-12+5)^2 + (-7+5)^2 + (-9+5)^2 + (5+5)^2}{9}$$

$$= \frac{4+81+16+4+144+64+49+4+16+100}{9}$$

$$= \frac{482}{9}$$

$$S_d^{\,2} = 53.55$$

$$S_d = 7.318$$

6. Decisior

-2.160 <

Example 5.12. Fin

a. $t_{0.025}$ when ν =

b. $-t_{0.01}$ when

Solution

a. $t_{0.025} = 2.179$

b. $-t_{0.01} = -2.896$

Example 5.13. Fin

a. $p(t < 2$

b. $p(t > 1.$

c. $p(-1.0$

d. $p(t > -$

Solution

a. When t < 2.306

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

$$= \frac{-5-0}{7.318/\sqrt{10}}$$

$$= \frac{-5}{7.318/3.162}$$

$$= \frac{-5}{2.314} = -2.160$$

6. Decision: $t < t_\alpha$ then null hypothesis is accept

   -2.160 < 2.821 null hypothesis is accepted

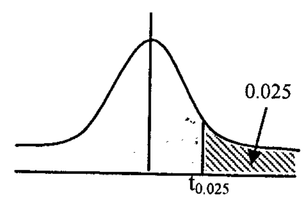
Example 5.12. Find

a. $t_{0.025}$ when $\nu = 12$

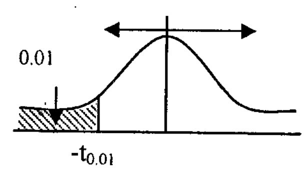
b. $-t_{0.01}$ when $\nu = 8$

Solution

a. $t_{0.025} = 2.179$



b. $-t_{0.01} = -2.896$



Example 5.13. Find

a. $p(t < 2.306)$ when $\nu = 8$

b. $p(t > 1.711)$ when $\nu = 24$

c. $p(-1.083 < t < 2.179)$ with $\nu = 12$

d. $p(t > -2.368)$ when $\nu = 17$

Solution

a. When $t < 2.306$

$p(t < 2.306)$ is given by the area to the left of t = 2.306

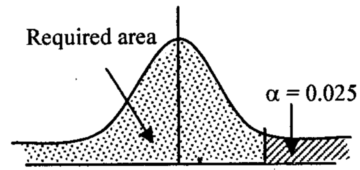
From table,

$t_\alpha = 2.306$ for $v = 8$ d.o.f then

$\alpha = 0.025$

$\therefore P(t < 2.306) = 1 - 0.025$

$= 0.975$ **Ans.**



b. When t > 1.711,

$p(t > 1.711)$ when $v = 24$, is given by the area to the right of t = 1.711

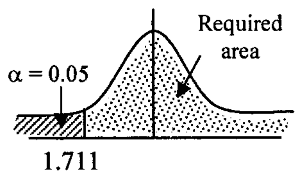
From table

$t_\alpha = 1.711$ for $v = 24$ d.o.f then

$\alpha = 0.05$

$p(t > 1.711) = 1 - 0.05$

$= 0.95$ **Ans.**



c. $p(-1.083 < t < 2.179)$ with $v = 12$

Area to the right of 2.179 is 0.025

$t > -1.083$ with $v = 12$,

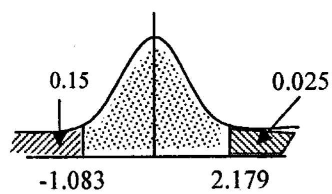
Area to the left is 0.15

When $-1.083 < t < 2.179$

The area is $1 - 0.15 - 0.025$

$= 0.825$

$P(-1.083 < t < 2.179) = 0.825$ **Ans.**



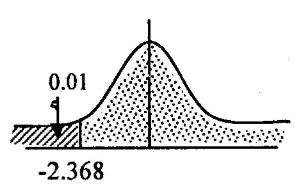
d. $p(t > -2.368)$ when $v = 17$

when $t > -2.368$

the area is $1 - 0.015$

$= 0.985$

$P(t > -2.308) = 0.985$ **Ans.**



Example 5.14. Twc
4.2 and 3.9 respectiv
than 0.05 justify your

Solution

Given that

Two independen
respectively

Example 5.15. Two
Find the probability th

Solution

Given that

Two random sam

$n_1 = 15$

$n_2 = 25$

$v_1 = n_1 - 1 = 15 -$

$v_2 = n_2 - 1 = 25 -$

$F_{0.01}(v_1, v_2) = F_{0.}$

$F_{0.05}(14, 24) = \frac{2.}{}$

So required proba

Example 5.16. Find
0.95 between the mean
standard deviations are

Solution

Given that

Example 5.14. Two independent random samples of sizes 8 and 7 gave variances 4.2 and 3.9 respectively. Do you think that such a difference has probability less than 0.05 justify your answer?

(Reg. April / May 2004 Set 4)

Solution

Given that

Two independent random samples of sizes 8 and 7, gave variances 4.2 and 3.9 respectively

$$n_1 = 8 \ \sigma_1^2 = 4.2$$

$$n_2 = 7 \ \sigma_2^2 = 3.9$$

$$\nu_1 = n_1 - 1 = 8 - 1 = 7$$

$$\nu_2 = n_2 - 1 = 7 - 1 = 6$$

$$F_{0.01}(\nu_1, \nu_2) = F_{0.01}(7, 6) = 8.26$$

$$F_{0.05}(\nu_1, \nu_2) = F_{0.05}(7, 6) = 4.21$$

Ans.

Example 5.15. Two random sample of sizes 15 ad 25 are taken from a N (μ, σ^2). Find the probability that the ratio of the sample variance does not exceed 2.28.

(Reg. April/April 2004 Set 3)

Solution

Given that

Two random samples of sizes 15 and 25

$n_1 = 15$

$n_2 = 25$

$\nu_1 = n_1 - 1 = 15 - 1 = 14$

$\nu_2 = n_2 - 1 = 25 - 1 = 24$

$F_{0.01}(\nu_1, \nu_2) = F_{0.01}(14, 24) = 2.96$

$F_{0.05}(14, 24) = \dfrac{2.18 + 2.11}{2} = 2.145$

So required probability is approximately equal to 0.05. **Ans.**

Example 5.16. Find the maximum difference that we can expect with probability 0.95 between the mean of samples of size 10 & 12 from a normal population if their standard deviations are found to be 2 and 3 respectively.

(Reg. April / May 2008 Set 2)

Solution

Given that

$$n_1 = 10, n_2 = 12, \alpha = \text{L.O.S.} = 5\%$$

Standard deviation $S_1 = 2$

$S_2 = 3$

$$\nu_1 = n_1 - 1, \quad \nu_2 = n_2 - 1$$
$$= 10 - 1 \quad\quad = 12 - 1$$
$$= 9 \quad\quad = 11$$

$$Sp^2 = \frac{n_1 s_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}$$

$$Sp^2 = \frac{10(2)^2 + 2(3)^2}{10 + 12 - 2}$$

$$= \frac{148}{20} = 7.4$$

$$Sp = 2.72$$

For small samples, t-distribution $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - \delta}{\sqrt{\dfrac{S_1 2}{n_1} + \dfrac{S_2^2}{n_2}}}$

$$t = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{Sp\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

So $\quad |\bar{x}_1 - \bar{x}_2| = t_{\alpha/2} . Sp . \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$

$t_{0.025}$ for 20 d.o.f. is 2.086

Then

$$|\bar{x}_1 - \bar{x}_2| = (2.086)(2.72)\sqrt{\frac{1}{10} + \frac{1}{12}} = 2.429 \quad\quad \textbf{Ans.}$$

Example 5.17. If two independent random samples of size $n_1 = 9$ and $n_2 = 16$ are taken from a normal population, what is the probability that the variance of the first sample will be at least 4 times as large as the variance of the second sample.

(Reg. April/May 2004)

(Reg. April / May 2004 Set 2)

Solution

Given that

$n_1 = 9$

$n_2 = 16$

$\nu_1 = n_1 - 1$

$= 9 - 1$

$= 8$

For the table

$F_{0.01}$ (ν_1, ν

We have the vari

of the second sam

F (ν_1,ν_2) =

For (A) and

Example 5.18.

0f 0.05) for ν_1 =

Solution

We have the

Example 5.19.

Solution

We have

$n_1 = 9$

$n_2 = 16$

$\nu_1 = n_1 - 1 \qquad \nu_2 = n_2 - 1$

$= 9 - 1 \qquad = 16 - 1$

$= 8 \qquad = 15$

For the table

$F_{0.01}(\nu_1, \nu_2) = F_{0.01}(8, 15) = 4.00 \qquad \ldots\ldots (A)$

We have the variance of the first sample will be at least 4 times as large as variance of the second sample

$$F(\nu_1, \nu_2) = \frac{S_1^2}{S_2^2} = \frac{4S_2^2}{S_2^2}$$

$F(\nu_1, \nu_2) = 4 \qquad \ldots\ldots (B)$

For (A) and (B) The desired probability is 0.01 **Ans.**

Example 5.18. Find the value of $F_{0.05}$ (corresponding to a left hand tail probability 0f 0.05) for $\nu_1 = 10$ and $\nu_2 = 24$ degrees of freedom.

Solution

We have the following result

$$F_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{F_\alpha(\nu_1, \nu_2)}$$

$$F_{0.95}(10, 24) = \frac{1}{F_{0.05}(24, 10)}$$

$$= \frac{1}{2.74} = 0.36496$$

Ans.

Example 5.19. Find the value of $F_{0.99}$ for 5 and 20 degrees of freedom

Solution

We have

$$F_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{F_\alpha(\nu_2, \nu_1)}$$

$$F_{0.99}(5, 20) = \frac{1}{0.01(20, 5)}$$

Ans.

9 and $n_2 = 16$ are
ariance of the first
id sample.
April/May 2004)
May 2004 Set 2)

$$= \frac{1}{4.56}$$
$$= 0.21929 \qquad \textbf{Ans.}$$

Example 5.20. For an F-distribution find

a. $F_{0.05}$ with $\nu_1 = 6$ and $\nu_2 = 15$

b. $F_{0.01}$ with $\nu_1 = 24$ and $\nu_2 = 30$

Solution

From the table

a. $F_{0.05}$ with $\nu_1 = 6$ and $\nu_2 = 15$ is 2.79 **Ans.**

b. $F_{0.01}$ with $\nu_1 = 24$ and $\nu_2 = 30$ is 2.47 **Ans.**

Example 5.21. From the following data find whether there is any significant liking in the habit of taking soft drinks among the categories of employees.

(Nov. 2006 Set)

Soft drinks	**Clerks**	**Teachers**	**Officers**
Pepsi	10	25	65
Thumps up	15	30	65
Fanta	50	60	30

Solution

Soft drinks	**Clerks**	**Teachers**	**Officers**	**Totals**
Pepsi	10	25	65	100
Thumps up	15	30	65	110
Fanta	50	60	30	140
Total	75	115	160	350

1. Null hypothesis $H_0 : p_1 = p_2 = p_3$ i.e. no difference
2. Alternative hypothesis $H_1 : p_1, p_2, p_3$ are not all equal i.e. there is Difference
3. Level of significance $\alpha = 0.05$
4. Critical region: $\aleph^2 < \aleph^2_\alpha$ then null hypothesis is accepted

$$\aleph^2_\alpha = \aleph^2_{0.05} = 5.991 \text{ for 2 degrees of freedom.}$$

5. Test of statistic

$$e_{ij} = \frac{n_j . x}{n} \text{ and } e_{2j} = \frac{n_j (n - x)}{n}$$

$$e_{11} = \frac{n_1 x}{n} = \frac{100 \times 75}{350} = 21.42$$

$$e_{12} = \frac{n_2\, x}{n} = \frac{100 \times 115}{350} = 32.85$$

$$e_{13} = \frac{n_3\, x}{n} = \frac{100 \times 160}{350} = 45.71$$

$$e_{2j} = \frac{n_j (n - x)}{n}$$

$$e_{21} = \frac{n_1 (n - x)}{n} = \frac{75 \times 110}{350} = 23.57$$

$$e_{22} = \frac{n_2 (n - x)}{n} = \frac{115 \times 110}{350} = 36.14$$

$$e_{23} = \frac{n_3 (n - x)}{n} = \frac{160 \times 110}{350} = 50.28$$

$$e_{31} = \frac{75 \times 140}{350} = 30 \qquad e_{32} = \frac{115 \times 140}{350} = 46 \qquad e_{33} = \frac{160 \times 140}{350} = 64$$

$$\aleph^2 = \sum_{i=1}^{3} \sum_{i=1}^{3} \frac{(O_{ij} - e_{ij})^2}{e_{ij}}$$

$$= \frac{(O_{11} - e_{11})^2}{e_{11}} + \frac{(O_{12} - e_{12})^2}{e_{12}} + \frac{(O_{13} - e_{13})^2}{e_{13}} + \frac{(O_{21} - e_{21})^2}{e_{21}} + \frac{(O_{22} - e_{22})^2}{e_{22}}$$

$$+ \frac{(O_{23} - e_{23})^2}{e_{23}} + \frac{(O_{31} - e_{31})^2}{e_{31}} + \frac{(O_{32} - e_{32})^2}{e_{32}} + \frac{(O_{33} - e_{33})^2}{e_{33}}$$

$$\aleph^2 = \frac{(10 - 21.42)^2}{21.42} + \frac{(25 - 32.85)^2}{82.85} + \frac{(65 - 45.71)^2}{45.71} + \frac{(15 - 23.57)^2}{25.57}$$

$$+\frac{(30-36.14)^2}{36.14}+\frac{(65-50.28)^2}{50.28}+\frac{(50-30)^2}{30}+\frac{(60-46)^2}{46}+\frac{(30-64)^2}{64}$$

$$=\frac{130.41}{21.42}+\frac{61.6225}{32.85}+\frac{372.104}{45.71}+\frac{73.44}{23.57}+\frac{37.6996}{36.14}+\frac{216.67}{50.28}$$

$$+\frac{400}{30}+\frac{196}{46}+\frac{1156}{64}$$

$$= 6.088+1.875+8.140+3.116+1.043+4.3094$$

$$+13.333+4.260+18.06$$

$$= 60.2244$$

6. Decision : $\aleph^2 < \aleph^2_\alpha$ then null hypothesis is accepted.

   60.2244 > 5.991 null hypothesis is rejected i.e. there is difference.

Example 5.22. A coin is tossed 60 times, 35 heads and 25 tails were observed. Test the hypothesis that the coin is fair.

(a) Find the $\aleph^2$ value and degrees of freedom do f.

(b) Test the hypothesis using a significance level of

(a) 0.05, (b) 0.10

Solution. Let 40 the null hypothesis that the coin is fair.

Face value	Head	Tail
Frequency	35	25

(a) The coin is tossed 60 times and there are 2 possible values. Head and tail, then the expected number of times each face comes is 30. Thus

$$x^2=\sum\frac{(O-E)^2}{E} = \frac{(35-30)^2}{30}+\frac{(25-30)^2}{30}$$

$$= \frac{25}{30}+\frac{25}{30}$$

$$= \frac{50}{30}$$

$$= 1.66$$

$$\aleph^2 = 1.66$$

Degree of freedom = dof = 2 – 1 = 1.

(b) Test the Hypothesis using a significance level of (a) 0.05, (b) 0.10

From the $\aleph^2$ – distribution table $\aleph^2$ value for dof = 1 at the $\alpha = 0.05$ significance level is c = 3.84. Since 1.66 < 3.84, null hypothesis is accepted that coin is fair. And at the $\alpha = 0.10$ significance level is C = 2.71. Since 1.66 < 2.71, null hypothesis is accepted that coin is fair.

Example 5.23. A die is tossed 60 times, observed face values given as following distribution table

Face value	1	2	3	4	5	6
Frequency	8	11	9	14	6	12

Test the hypothesis that the die is fair

(a) Find the $\aleph^2$ value and degrees of freedom dof

(b) Test the hypothesis using a significance level of (a) 0.05, (b) 0.10

Solution

(a) The die is tossed 60 times and there are 6 possible face values. Then the expected number of times each face comes is 10. Thus

$$x^2 = \sum \frac{(0-E)^2}{E}$$

$$= \frac{(8-10)^2}{10} + \frac{(11-10)^2}{10} + \frac{(9-10)^2}{10} + \frac{(14-10)^2}{10} + \frac{(6-10)^2}{10} + \frac{(12-10)^2}{10}$$

$$= \frac{4}{10} + \frac{1}{10} + \frac{1}{10} + \frac{16}{10} + \frac{16}{10} + \frac{4}{10}$$

$$= \frac{42}{10}$$

$$x^2 = 4.2$$

$\aleph^2 = 4.2$

Degrees of freedom dof = 6 – 1 = 5.

(b) Test the Hypothesis using a significance level of (a) 0.05, (b) 0.10

From the $\aleph^2$ – distribution table $\aleph^2$ value for dof = 5 at the $\alpha = 0.05$ significance level is C = 11.07. Since 4.2 < 11.07, null hypothesis is accepted that die is fair and at the $\alpha = 0.10$, significance level is C = 9.24. Since 4.2 < 9.24 null hypothesis is accepted that die is fair.

Goodness of fit for Binomial Distribution

Example 5.24. There are 4 engineering colleges A, B, C, and D in a city. A group of 600 students tried to take admission in different colleges, the following number of colleges visited by each student.

Number of colleges	0	1	2	3	4
Number of students	130	170	50	10	240

If P = 0.60, test the hypothesis at the $\alpha = 0.10$ significance level that the distribution is binomial.

Solution

Let H_0 be the null hypothesis that the distribution is binomial

The binomial distribution with n = 4 and p = 0.60.

$$p(0) = b(x; n, p) = b(0; 4, 0.60) = 4C_0(0.60)^0(0.40)^{4.0} = (0.40)^4 =$$
$$p(1) = b(x; n, p) = b(1; 4, 0.60) = 4C_1(0.60)^1(0.40)^3 = 4(0.60)(0.40)^3 =$$
$$p(2) = b(x; n, p) = b(2; 4, 0.60) = 4C_2(0.60)^2(0.40)^2 = 6(0.60)^2(0.40)^2 =$$
$$p(3) = b(x; n, p) = b(3; 4, 0.60) = 4C_3(0.60)^3(0.40)^1 = 4(0.60)^3(0.40)^1 =$$
$$p(4) = b(x; n, p) = b(4; 4, 0.60) = 4C_4(0.60)^4(0.40)^0 = (0.60)^4 =$$

$p(0) = 0.0256$

$p(1) = 0.1536$

$p(2) = 0.3456$

$p(3) = 0.3456$

$p(4) = 0.1296$

Multiplying the probability by the number of students (600) gives expected value

Number of colleges	0	1	2	3	4
Expected number of students	15.36	92.16	207.36	207.36	77.76

$$\aleph^2 = \sum \frac{(O-E)^2}{E}$$
$$= \frac{(130-15.36)^2}{15.36} + \frac{(170-92.16)^2}{92.16} + \frac{(50-207.36)^2}{207.36} + \frac{(10-207.36)^2}{207.36} + \frac{(240-7}{77.7}$$
$$= 855.62 + 97.47 = 119.41 + 187.84 + 338.50$$
$$= 1598.840$$

dof = 5-1=4

From $\aleph^2$

From $\aleph^2$ – dist

1598.840>7.78, \

with P = 0.60.

$E =$

M.E. for Small S

1) A random
observed tl
mean is 10(

[Hint: maxin

$S^2 = \sum_{i=1}^{n}$

$E = t_{\alpha/2}\, S$

2) A sample o
is 0.03 find

[Hint: $E =$

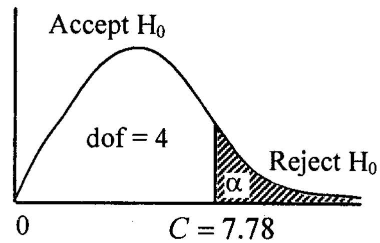
Confidence Inter

3) If standard
Confidence

[Hint: Conf

dof = 5-1=4

From $\aleph^2$ – distribution table $\alpha = 0.10$ for dof 4, critical value c = 7.78.



From $\aleph^2$ – distribution table α = 0.10 for dof 4, critical value c = 7.78.Since 1598.840>7.78, we reject the null hypothesis H_0, that the distribution is binomial with P = 0.60.

$$E = t_{\alpha/2}\ S/\sqrt{n}\ \bar{x} - t_{\alpha/2}\, S/\sqrt{n} < \mu < \bar{x} + t_{\alpha/2}\ S/\sqrt{n}$$

EXERCISE

M.E. for Small Sample

1) A random sample of size 10, is taken from a normal population. It is observed that Mean is 21.5 and the sum of the squares of deviations from mean is 100. find the maximum error with 95% confidence.

[**Hint**: maximum error for the small samples $E = t_{\alpha/2}\ S/\sqrt{n}$

$$S^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1} = 100/(10\text{-}1) = 11.11,\ S=3.33,\ t_{\alpha/2} = 2.262$$

$E = t_{\alpha/2}\ S/\sqrt{n}$ = 2.262 ×3.33/√10 = 2.382 **Ans.**]

2) A sample of size 15 is taken from a population; standard deviation of sample is 0.03 find the maximum error with 99% confidence.

[**Hint**: $E = t_{\alpha/2}\ S/\sqrt{n}$ =2.977×0.03/√15=2.977×0.03/3.872=0.02306 **Ans.**]

Confidence Interval for small sample

3) If standard deviation of the sample is 10, sample size n=20, construct 95% Confidence Interval if $\bar{x}$ = 50.

[**Hint**: Confidence Interval = $(\bar{x} - t_{\alpha/2}\,\sigma/\sqrt{n} < \mu < \bar{x} + t_{\alpha/2}\ \sigma/\sqrt{n})$

= (50-2.093× (10/√20), 50+2.093× (10/√20)) = (45.3197, 54.6802) **Ans.**]

Hypothesis concerning one mean small sample

4) In a college 10 students were randomly selected. It is found that their marks in Mathematics is 20,22,21,15,25,19,18,22,21,20. Test the hypothesis that on an Average the students get 25 marks.

[Hint: N.H: Ho: $\mu=25$

A.H: H_1 $\mu\neq 25$

L.O.S $\alpha = 0.05$

Test statistic $t = \dfrac{\bar{x}-\mu}{S/\sqrt{n}}$

$\bar{X} = 20.3$

$$S^2 = \sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{n-1} = 6, \quad S = 2.45$$

t= (20.3-25)/ (2.45/√10)

= -5.59

$t = -5.59 < t_{\alpha/2} = 2.26$, Ho rejected **Ans.]**

5) Two independent samples of 7 items had the following values.

Sample I	14	12	9	15	11	13	11
Sample II	10	8	9	13	10	11	9

is the difference between the means of samples significant?

[Hint: N.H Ho: $\mu_1 = \mu_2$

A.H H1: $\mu_1 \neq \mu_2$

L.O.S = 0.05

Test statistic $z = \bar{x}-\bar{y} \Big/ (S\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}})$

$\bar{x} = 85/7 = 12.142, \quad \bar{y} = 70/7 = 10$

$$S^2 = \frac{\sum(x_i-\bar{x})^2 + \sum(y_i-\bar{y})^2}{n_1+n_2-2} = \frac{24.856+16}{7+7-2} = 3.4046$$

S = 1.845

t = 12.142-10/3.4046√1/7+1/7 = 1.7879

$t = 1.7879 < t_{\alpha/2} = 2.179$, Ho is accepted]

(Before and after kind of data)

6) In a study of effective ness of teachers training programme in J.N.T.U Hyderabad, A group of 10 teachers attended the above programme and the number which they got in the feedback which is taken in the classroom before and after training are as follows

No Before	9
No After	9

Use 0.05 level
programme is e

[Hint: N.H Ho: μ_1=
A.H H1: μ_1
L.O.S = 0.

Test statistic t = -

Differences d_i -2 -10

$\bar{x}$ = [-2+(-

$(d_i-\bar{x})^2$ = [-2-(-4)]²+
[-3-(-4)]

S = $(d_i-\bar{x})^2$/

t = (-4-0)/(6/

t_α = 1.833 for v = 9

(Small samples Diffe

7) Find the standa
0.05 level of sig

Sam
Sam

[Hint: S.E =√(S
= √

Confidence inter

F-Distribution

8) Two independent

Sample I
Sample I

Test at 5% significa

No Before	90	80	85	83	75	77	82	92	90	85
No After	92	90	90	85	80	80	85	95	92	90

Use 0.05 level of significance to test whether the prescribed teachers training programme is effective.

[**Hint**: N.H Ho: $\mu_1 = \mu_2$
A.H H1: $\mu_1 > \mu_2$
L.O.S = 0.05

Test statistic $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$

Differences d_i -2 -10 -5 -2 -5 -3 -3 -3 -2 -5

$\bar{x}$ = [-2+(-10)+ (-5)+ (-2)+ (-5)+ (-3)+ (-3)+(-3)+(-2)]+(-5)/10 = -40/10 = -4

$(d_i-\bar{x})^2 = [-2-(-4)]^2 + [-10-(-4)]^2 + [-5-(-4)]^2 + [-2-(-4)]^2 + [-5-(-4)]^2 + [-3-(-4)]^2 + [-3-(-4)]^2 + [-3-(-4)]^2 + [-2-(-4)]^2 + [-5-(-4)]^2 = 4+36+1+4+1+1+1+1+4+1 = 54$

$S = (d_i-\bar{x})^2/(n-1) = 54/9 = 6$

$t = (-4-0)/(6/\sqrt{10}) = -4/1.8974 = -2.108$

$t_{\alpha = 1.833}$ for $v = 9$, $t = 2.108 > t_\alpha = 1.833$ Ho is rejected.]

(Small samples Difference between the mean)

7) Find the standard error and the confidence interval for difference the mean at 0.05 level of significance for the following data

	Size	Mean	Standard deviation
Sample I	15	37	5
Sample II	20	26	8

[**Hint**: S.E $=\sqrt{(S_1^2/n_1 + S_2^2/n_2)}$
$= \sqrt{(5^2/15) + (8^2/20)} = 2.2059$

Confidence interval $= \bar{x} - \bar{y} \pm t_{\alpha/2}\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}$ $= 37-26 \pm \sqrt{(5^2/15)+(8^2/20)}$

$= 11 \pm (2.2059)$

F-Distribution

8) Two independent samples of 8 items have the following values

Sample I	10	11	12	13	15	10	11	9
Sample II	8	9	10	11	13	15	10	8

Test at 5% significance level, the quality of variances of two population

[**Hint:** Null Hypothesis Ho: $S_1^2 = S_2^2$
A.H H_1: $S_1^2 \neq S_2^2$
L.O.S α=0.05

Test statistic $F = \dfrac{S_1^2}{S_2^2}$

$\bar{x}$ = 91/8=11.375, $\bar{y}$ = 84/8 = 10.5

$S_1^2 = \sum_{i=1}^{n} \dfrac{(x_i - \bar{x})^2}{n_1 - 1}$ = 3.695, S_1 = 1.922 ,S_2^2 = 42/7 = 6, S_2 = 2.449

$F = \dfrac{S_1^2}{S_2^2}$ =3.695/6=0.6158

F_2 (7, 7) =3.79, F=0.6158< F_2 (7, 7) =3.79, N.H is accepted, No difference]

OBJECTIVE TYPE QUESTIONS

1. $F_{1-\alpha}(\upsilon_1, \upsilon_2) =$

(a) $F_\alpha(\upsilon_2, \upsilon_1)$ (b) $1 / F_\alpha(\upsilon_1, \upsilon_2)$ (c) $F_\alpha(\upsilon_1, \upsilon_2)$ (d) none of these

Ans. (b)

2. If p = 0.50 the maximum error with 95% confidence is 0.08 then the sample size is

(a) 155 [illegible]0 (c) 158 (d) None of the above

Ans. (b)

3. Ten students participated in car racing competition in the first week,Afte intensive training they participated in the same competition in the next week. Their scores before and after training are given as follows

Scores Before	67	24	57	55	63	54	56	68	33	43
Scores After	70	38	58	58	56	67	68	75	42	38

What is the mean of the scores before and after training?

(a) 52, 55 (b) 55, 57 (c) 52, 57 (d) None of the above

Ans. (c)

4. If p = 0.60 th
size is

(a) 13

Ans. (d)

5. For the samp
confidence if

(a) 0.0125

Ans. (b)

6. For the small s
estimate

(a) $E = t_{\alpha/2}$

(c) $E = t_{\alpha/2}$

Ans. (d)

7. From $\aleph^2$-distr

(a) 3.23 & 32

(c) 4.18 & 50

Ans. (b)

8. From F- distrib

(a) 1.63 & 2.8

(c) 1.63 & 1.6

Ans. (a)

9. From $\aleph^2$-distr

(a) 11.34

Ans. (a)

10. From F- distri

(a) 0.222

Ans. (c)

4. If p = 0.60 the maximum error with 95% confidence is 0.25 then the sample size is

   (a) 13 (b) 14 (c) 12 (d) 15

Ans. (d)

5. For the sample size 10, what will be the maximum error with 99% confidence if the standard deviation of sample is 0.03?

   (a) 0.0125 (b) 0.0325 (c) 0.025 (d) None of the above

Ans. (b)

6. For the small samples if σ is unknown or (n < 30), The maximum error estimate

   (a) $E = t_{\alpha/2}\ S^2/\sqrt{n}$ (b) $E = Z_{\alpha/2}\ S/\sqrt{n}$

   (c) $E = t_{\alpha/2}\ S/\sqrt{N}$ (d) $E = t_{\alpha/2}\ S/\sqrt{n}$

Ans. (d)

7. From $\aleph^2$-distribution table, the values of $\aleph^2_{0.95,\,8}$ & $\aleph^2_{0.025,\,20}$ are

   (a) 3.23 & 32 (b) 2.73 & 34.17

   (c) 4.18 & 50 (d) None of the above

Ans. (b)

8. From F- distribution table, the values of $F_{0.25,4,9}$ & $F_{0.05,15,10}$ is

   (a) 1.63 & 2.85 (b) 2.85 & 2.85

   (c) 1.63 & 1.63 (d) None of the above

Ans. (a)

9. From $\aleph^2$-distribution table, the values of $\aleph^2_{0.50,\,12}$ is

   (a) 11.34 (b) 12.50 (c) 9.56 (d) None of the above

Ans. (a)

10. From F- distribution table, the values of $F_{0.90,24,24}$ is

   (a) 0.222 (b) 0.235 (c) 0.588 (d) None of the above

Ans. (c)

11. From t- distribution table, the values of $t_{0.25,10}$ is

(a) 0.333 (b) 0.700 (c) 0.588 (d) None of the above

Ans. (b)

12. From t- distribution table, the values of $t_{0.25,20}$ is

(a) 0.555 (b) 0.700 (c) 0.687 (d) None of the above

Ans. (c)

13. Two samples which is obtained from normal population if $S_1^2 = 3.37, S_2^2 = 0.46$ Then F is

(a) 8.5 (b) 8.3 (c) 7.3 (d) None of the above

Ans. (c)

14. If for two samples $S_1^2 = 666.67, S_2^2 = 1109.33$ Then F is

(a) 1.66 (b) 3.25 (c) 1.25 (d) None of the above

Ans. (a)

15. From F- distribution table, the values of $F_{0.01,15,9}$ is

(a) 0.257 (b) 0.235 (c) 0.588 (d) None of the above

Ans. (a)

ıe of the above

ıe of the above

ıe of the above

ıe of the above

ıe of the above

UNIT-6

CORRELATION & REGRESSION

"Knowledge speaks, but wisdom listens"

6.1 FITTING OF STRAIGHT LINE

The least square line that fits a set of sample points is given by

$$y = a + bx$$

Where the constants a and b are determined by the normal equations

$$\sum y = na + b\sum x \text{..........}(1)$$

$$\sum xy = a\sum x + b\sum x^2 \text{..........}(2)$$

Or the constants a and b can be obtained by

$$a = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n\sum x^2 - (\sum x)^2},\ b = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

b can also be written as

$$b = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}$$

Where $\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$

Or the constants a and b can be obtained by

$$S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 \Big/ n$$

$$S_{yy} = \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2 \Big/ n$$

$$S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n$$

$$a = \bar{y} - b\bar{x} \quad \text{And } b = \frac{S_{xy}}{S_{xx}}$$

Working Rule for the Fitting of straight line

$$y = a + bx$$

1. Consider the set of given data $(x_i, y_i), i = 1, 2, \ldots\ldots n$
2. Find Σx, Σy, Σxy, Σx^2, n from the given data.
3. Substitute these values in the normal equations.
4. Solve these normal equations for a & b.
5. Put these values in $y = a + bx$, which is the required curve of best fit.

6.2 CURVILINEAR REGRESSION

6.2.1 Fitting of Second Degree Polynomial

The least square parabola that fits a set of sample points is given by

$$y = a + bx + cx^2$$

Where the constants a, b, c, are determined by solving the normal equations.

$$\sum y = na + b\sum x + c\sum x^2 \ldots\ldots\ldots.(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \ldots\ldots.(2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \ldots\ldots.(3)$$

Working Rule for the Fitting of Second Degree Polynomial

$$y = a + bx + cx^2$$

1. Consider the set of given data $(x_i, y_i), i = 1, 2, \ldots\ldots n$
2. Find Σx, Σy, Σxy, $\Sigma x^2 y$, Σx^2, Σx^3, Σx^4, n from the given data.
3. Substitute these values in the normal equations.
4. Solve these normal equations for a, b & c
5. Put these values in $y = a + bx + cx^2$, which is the required curve of best fit.

6.2.2 Fitting of Geon

Non linear cur

$y = ax^b$

By taking logari

Which is the cu

Where

Y

The constants A

$\sum Y = n$

$\sum XY =$

Working Rule

1. Find Y =
2. Find ΣY,
3. Substitute
4. Solve the
5. Obtain a
6. Put these

6.2.3 $y = ab^x$ **(Ex**

Taking logarit

Which is the c

Where $Y = \mathrm{lc}$

Normal equati

The constants

Working Rul

1. Find Y =
2. Find ΣY,

6.2.2 Fitting of Geometric Curve or Power Curve

Non linear curves can be transformed to a linear curve straight line consider

$y = ax^b$

By taking logarithms on both sides we obtain

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

or

$$Y = A + BX$$

Which is the curve of straight line

Where

$$Y = \log_{10} y,\ A = \log_{10} a, X = \log_{10} x,\ b = B$$

The constants A and B are determined by the normal equations

$$\sum Y = nA + B\sum X ....(1)$$

$$\sum XY = A\sum X + B\sum X^2 .....(2)$$

Working Rule for the fitting of Geometric Curve $y = ax^b$

1. Find Y = log y and X = log x for all the data.
2. Find ΣY, ΣX, ΣXY, ΣX^2
3. Substitute all these values in the normal equations.
4. Solve these normal equations for A and B
5. Obtain a by taking antilog of A and b = B.
6. Put these values in $y = ax^b$ which is the required curve of best fit.

6.2.3 $y = ab^x$ (Exponential Curve)

Taking logarithms on both sides we obtain $\log_{10} y = \log_{10} a + x \log_{10} b$

$$Y = A + XB$$

Which is the curve of straight line

Where $Y = \log y, A = \log a, X = x, B = \log b$

Normal equations are similar to a straight line.

The constants A and B are determined by the normal equations

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

Working Rule for the fitting of exponential curve $y = ab^x$

1. Find Y = log y and X = x for all the data.
2. Find ΣY, ΣX, ΣXY, ΣX^2

3. Substitute all these values in the normal equations.
4. Solve these normal equations for A and B
5. Obtain a and b, by taking antilog of A and B.
6. Put these values in $y = ab^x$. Which is the required curve of best fit?

6.2.4 $y = ae^{bx}$ (Exponential Curve)

Taking logarithms on both sides we obtained

$$\log_{10} y = \log_{10} a + bxog_{10}e$$

$$Y = A + XB$$

Which is the curve of straight line

Where

$$Y = \log y, A = \log a,, B = b, X = x\log_{10} e$$

Normal equations are similar to a straight line.

The constants A and B are determined by the normal equations.

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

Working Rule for the fitting of exponential curve $y = ae^{bx}$

1. Find Y = log y and X = x log e for all the data.
2. Find ΣY, ΣX, ΣXY, ΣX^2
3. Substitute all these values in the normal equations.
4. Solve these normal equations for A and B
5. Obtain a by taking antilog of A and b = B.
6. Put these values in $y = ae^{bx}$. Which is the required curve of best fit?

6.2.5 $y = \dfrac{1}{a_0 + a_1 x}$ (Hyperbola) OR (Reciprocal Function)

$$\frac{1}{y} = a_0 + a_1 x$$

$$Y = a_0 + a_1 X$$

Which is the curve of straight line?

Where $Y = \dfrac{1}{y}, X = x$

Normal equations are similar to a straight line

$$\Sigma Y = na_0 + a_1 \Sigma X$$

$$\Sigma XY = a_0 \Sigma X + a_1 \Sigma X^2$$

Working

1. Find
2. Find Σ
3. Substi
4. Solve
5. Put th

6.3 MULTIPI

If there is independent

This is call

The consta

$\sum y =$

$\sum x_1 y$

$\sum x_2 y$

Working R

1. Consid
2. Find Σ
3. Substit
4. Solve t
5. Put the best fit

6.4 RESIDUAL

Observation

The minimu or error sum

Working Rule for the fitting of Reciprocal function $y = \dfrac{1}{a_0 + a_1 x}$

1. Find $Y = \dfrac{1}{y}$, X = x for all the data.
2. Find $\Sigma Y, \Sigma X, \Sigma XY, \Sigma X^2$
3. Substitute all these values in the normal equations.
4. Solve these normal equations for a_0 and a_1.
5. Put these values in $y = \dfrac{1}{a_0 + a_1 x}$. Which is the required curve of best fit?

6.3 MULTIPLE REGRESSION

If there is a linear relationship between a dependent variable y and two independent variables x_1 and x_2, then

$$y = a_0 + a_1 x_1 + a_2 x_2$$

This is called a regression equation of y on x_1 and x_2.

The constants a_0, a_1 and a_2 are determined by the normal equations.

$$\sum y = na_0 + a_1 \sum x_1 + a_2 \sum x_2 \text{............}(1)$$

$$\sum x_1 y = a_0 \sum x_1 + a_1 \sum x_1^2 + a_2 \sum x_1 x_2 \text{........}(2)$$

$$\sum x_2 y = a_0 \sum x_2 + a_1 \sum x_1 x_2 + a_2 \sum x_2^2 \text{.........}(3)$$

Working Rule for the fitting of multiple regression equation

$$y = a_0 + a_1 x_1 + a_2 x_2$$

1. Consider the set of given data (x_i, y_i), i = 1, 2, ...n.
2. Find Σy, Σx_1, Σx_2, $\Sigma x_1 y$, Σx_1^2, $\Sigma x_1 y_2$, $\Sigma x_2 y$, $\Sigma x_2^2$from the given data.
3. Substitute all these values in the normal equations.
4. Solve these normal equations for a_0, a_1, & a_2
5. Put these values in $y = a_0 + a_1 x_1 + a_2 x_2$, which is the required curve of best fit?

6.4 RESIDUALS

Observation – filled value = $y_i - a - bx_i$

The minimum value of the sum of squares is called the residual sum of squares or error sum of squares.

6.4.2 Residual sum of squares

$$= \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

$$= S_{yy} - S_{xy}^{\ 2} / S_{xx}$$

6.4.3. Correlation Coefficient r

$$r = \frac{1}{n-1} \sum_{i=1}^{n} \left(\frac{x_1 - \bar{x}}{S_x} \right) \left(\frac{y_i - \bar{y}}{S_y} \right)$$

Or

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

Or

$$r = \frac{\sqrt{S_{xx}}}{\sqrt{S_{yy}}} = b$$

Or

$$r = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\Sigma (x_i - \bar{x})^2 (\Sigma y_i - \bar{y})^2}}$$

Or

$$r = \frac{n \Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{\left[n \Sigma x^2 (\Sigma x)^2 \right] \left[n \Sigma y^2 - (\Sigma y)^2 \right]}}$$

6.4.4 Standard error of estimate

$$S_e^2 = \frac{1}{n-2} \sum_{i=1}^{n} \left[y_1 - (a + bx_i) \right]^2$$

$$= \frac{S_{yy} - (S_{xy})^2 / S_{xx}}{n-2}$$

With (n – 2) degree of freedom

6.4.5 The statistics – The values of random variables having t – Distribution is

With

6.4.6 Confiden

α:

β:

6.5 SPEARMA

For a give

ranked in order

observations x an

The Spearman's r

r_{rank} or

Where d →

n →

6.6 THE LINES

6.6.1. The line of

6.6.2. The line of

$$t = \frac{(a-\alpha)}{Se}\sqrt{\frac{nS_{xx}}{S_{xx}+n(\bar{x})^2}}$$

and

$$t = \frac{(b-B)}{Se}\sqrt{S_{xx}}$$

With (n – 2) degree of freedom

6.4.6 Confidence intervals for the regression coefficients α and β.

$$\alpha:\ a \pm t_{\alpha/2}.Se\ \sqrt{\frac{1}{n}+\frac{(\bar{x})^2}{S_{xx}}}$$

$$\beta:\ b \pm t_{\alpha/2}.Se\ \sqrt{\frac{1}{S_{xx}}}$$

6.5 SPEARMAN'S RANK CORRELATION COEFFICIENT

For a given set of n paired observations (x_i, y_i) for i = 1 to n. The data may be ranked in order of size, using the number 1, 2, …….n. If given set of n paired observations x and y (x_i, y_i) are ranked in such manner.

The Spearman's rank correlation coefficient denoted by

r_{rank} or r is given by

$$r_{rank} = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

Where d → difference between ranks of corresponding x and y.

n → number of pairs of data

6.6 THE LINES OF REGRESSION

6.6.1. The line of regression of y on x

$$y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

6.6.2. The line of regression of x on y

$$x-\bar{x} = r\,\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

Where $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}, \bar{y} = \dfrac{\sum_{i=1}^{n} y_i}{n}$

$$\sigma_x^2 = \frac{1}{n}\sum x^2 - \bar{x}^2, \qquad \sigma_y^2 = \frac{1}{n}\sum y^2 - \bar{y}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2, \qquad = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

6.7 Theorem Derive normal equations to fit the straight line.

Proof

Suppose a given set of n data points $(x_1, y_1)(x_2, y_2)......(x_n.y_n)$ and it is required to fit a straight line $y = a + bx$

For each x_i, expected value is $a + bx_i$

Error =Observed Value – Expected Value

$$e_i = y_i - (a + bx_i)$$

$$e_i = \sum_{i=1}^{n}(y_i - a - bx_i)$$

The sum of the squares of these errors is

$$E = e_1^2 + e_2^2 + ... + e_n^2$$

$$= \sum_{i=1}^{n}(y_1 - a - bx_i)^2$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 \quad and \quad \frac{\partial E}{\partial b} = 0$$

$$\Rightarrow 2\sum_{i=1}^{n}[y_1 - (a + bx_i)](-1) = 0$$

$$\Rightarrow 2\sum_{i=1}^{n}[y_1 - (a + bx_i)](-x_i) = 0$$

On rewriting the above equations

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$

These linear

unknowns a

6.8 Theorem Deri

Derive norm

Proof

Consider the

NOTE: Secc

curvilinear re

We know tha

e_i

The sum of th

$E = [y_1 - (a + bx_1$

For E to be minimun

$\Rightarrow 2[y_1 - (a + bx_1$

$\Rightarrow 2[y_1 - (a + bx_1$

Similarly

$\Rightarrow 2[y_1 - (a + bx_1$

These linear equations known as normal equations and can be solved for unknowns a & b.

6.8 Theorem Derive normal equations to fit the parabola

$$y = a_0 + a_1 x + a_2 x^2$$

Derive normal equations to fit the parabola

$$y = a + bx + cx^2$$

(Regular April/ May 2005 Set No.4)
(Supple Nov. /Dec 2005 Set No.4)
(Supple. Feb 2007 Set No.3)
(Supple.Feb.2010Set No.3)

Proof

Consider the equation of parabola or second degree polynomial

$$y = a + bx + cx^2$$

NOTE: Second degree polynomial is the example of non linear regression or curvilinear regression

We know that

$$\text{Error} = \text{observed value} - \text{expected value}$$

$$e_i = y_i - \left(a + bx_i + cx_i^{\,2}\right) \qquad \forall\, i = 1,2,....n$$

The sum of the squares of the errors is

$$E = e_1^2 + e_2^2 + ... + e_n^2$$

$$E = \left[y_1 - \left(a + bx_1 + cx_1^2\right)\right]^2 + \left[y_2 - \left(a + bx_2 + cx_2^2\right)\right]^2 + ... + \left[y_n - \left(a + bx_n + cx_n^2\right)\right]^2$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0$$

$$\Rightarrow 2\left[y_1 - \left(a + bx_1 + cx_1^2\right)\right](-1) + 2\left[y_2 - \left(a + bx_2 + cx_2^2\right)\right](-1) + ...$$
$$+ 2\left[y_n - \left(a + bx_n + cx_n^2\right)\right] = 0 ....(1)$$

$$\frac{\partial E}{\partial b} = 0$$

$$\Rightarrow 2\left[y_1 - \left(a + bx_1 + cx_1^2\right)\right](-x_1) + 2\left[y_2 - \left(a + bx_2 + cx_2^2\right)\right](-x_2) + ...$$
$$+ 2\left[y_n - \left(a + bx_n + cx_n^2\right)(-x_n)\right] = 0 ....(2)$$

Similarly

$$\frac{\partial E}{\partial c} = 0$$

$$\Rightarrow 2\left[y_1 - \left(a + bx_1 + cx_1^2\right)\right](-x_1^2) + 2\left[y_2 - \left(a + bx_2 + cx_2^2\right)\right](-x_2^2) + ...$$
$$+ 2\left[y_n - \left(a + bx_n + cx_n^2\right)(-x_n^2)\right] = 0 ....(3)$$

On rewriting the equations (1), (2) and (3)

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3$$

$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4$$

These linear equations know as normal equations and can be solved for unknowns a, b and c.

6.9 MULTIPLE REGRESSION

Consider the curve

$$y = a_0 + a_1x_1 + a_2x_2 + \dots\dots + a_nx_n$$

Where $x_1x_2\dots\dots x_n$ n independent variables and y are is a dependent variable.

Now we consider, y is depend on two independent variable x_1 and x_2.

$$y = a_0 + a_1x_1 + a_2x_2$$

Error = observed value -- expected value

$$e_i = \sum_{i=1}^{n}\left[y_i - (a_0 + a_1x_{i1} + a_2x_{i2})^2\right] \qquad \forall i = 1,2,\dots.n$$

The sum of the squares of the errors is

$$E = e_1^2 + e_2^2 + \dots\dots + e_n^2$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a_0} = 0,\ \frac{\partial E}{\partial a_1} = 0 \ \ and\ \ \frac{\partial E}{\partial a_2} = 0$$

On differentiating and simplifying, we obtain

$$\sum y = na_0 + a_1\sum x_1 + a_2\sum x_2$$

$$\sum x_1 y = a_0\sum x_1 + a_1\sum x_1^2 + a_2\sum x_1x_2$$

$$\sum x_2 y = a_0\sum x_2 + a_1\sum x_1x_2 + a_2\sum x_2^2$$

These linear equations are the normal equations and it can be solved for unknown's a_0, a_1 and a_2.

SOLVED EXAMPLES

Example 6.1. The following data pertain to the demand for a product (In thousands of units) and its price (in cents) charged in five different marked areas.

Price (x)	20	16	10	11	14

Fit a parabola

Solution

Price (x)	Demand (y)
20	22
16	41
10	120
11	89
14	56
Σx = 71	Σy = 328

$y = a_0 + a_1x$

Normal equatic

$\Sigma y = na_0+a_1\Sigma x$

$\Sigma xy = a_0\Sigma x + a$

$\Sigma x^2y = a_0\Sigma x^2+$

53(

Eq (1) divided

Eq (2) divided

$a_($

Eq (3) divided by 1(

$a_0 +$

From (4) and (5)

$a_0 + 14.2a_1 + 214.$

Demand(y)	22	41	120	89	56

Fit a parabola of the form $y = a_0 + a_1x + a_2x^2$ to the above data.

(Nov./Dec.2004 Set 4)

Solution

Price (x)	Demand (y)	xy	x^2	x^2y	x^3	x^4
20	22	440	400	8800	8000	160000
16	41	656	256	10496	4096	65536
10	120	1200	100	12000	1000	10000
11	89	979	121	10769	1331	14641
14	56	784	196	10976	2744	38416
$\Sigma x = 71$	$\Sigma y = 328$	$\Sigma xy = 4059$	$\Sigma x^2 = 1073$	$\Sigma x^2y = 53041$	$\Sigma x^3 = 17171$	$\Sigma x^4 = 288593$

$$y = a_0 + a_1x + a_2x^2$$

Normal equations are

$$\Sigma y = na_0 + a_1\Sigma x + a_2\Sigma x^2$$

$$\Sigma xy = a_0\Sigma x + a_1\Sigma x^2 + a_2\Sigma x^3$$

$$\Sigma x^2y = a_0\Sigma x^2 + a_1\Sigma x^3 + a_2\Sigma x^4$$

$$328 = 5a_0 + 71a_1 + 1073a_2 \ldots\ldots\ldots\ldots(1)$$

$$4059 = 71a_0 + 1073a_1 + 17171a_2 \ldots\ldots\ldots(2)$$

$$53041 = 1073a_0 + 17171a_1 + 288593a_2 \ldots\ldots\ldots(3)$$

Eq (1) divided by 5

$$a_0 + 14.2a_1 + 214.6a_2 = 65.6 \ldots\ldots\ldots\ldots(4)$$

Eq (2) divided by 71

$$a_0 + 15.11a_1 + 241.84a_2 = 57.16 \ldots\ldots\ldots\ldots(5)$$

Eq (3) divided by 1073

$$a_0 + 16.0027a_1 + 268.95a_2 = 49.432 \ldots\ldots\ldots\ldots(6)$$

From (4) and (5)

$$a_0 + 14.2a_1 + 214.6a_2 = 65.6$$

$$\underline{a_0 + 15.11a_1 + 241.84a_2 = 57.16}$$

$$-0.91_1 - 27.24a2 = 8.44$$

$$0.91a_1 + 27.24\ a_2 = -\ 8.44$$

Divided by 0.91

$$a_1 + 29.89a_2 = -\ 9.274 \ \ldots\ldots.. (7)$$

From (5) and (6)

$$a_0 + 15.11a_1 + 214.84a_2 = 57.16$$
$$\underline{a_0 + 16.0027a_1 + 268.95a_2 = 49.432}$$

$$-0.89a_1\ - 27.11a2\ =\ 7.728$$

$$0.89a_1 + 27.11\ a_2 = -\ 7.728$$

Divided by 0.89

$a_1 + 30.46a_2 = -\ 8.683 \ \ldots\ldots.. (8)$

From (7) and (8)

$a_1 + 29.89a_2 = -\ 9.274$

$\underline{a_1 + 30.46a_2 = -\ 8.683}$

$-\ 0.57\ a_2 = -\ 0.591$

$a_2 = 1.036$

$a_1 + 29.89a_2 = -\ 9.274$

$a_1 = -\ 9.274 - 29.89\ (1.036)$

$a_1 = -\ 9.274 - 30.99$

$a_1 = -\ 40.265$

$a_0 + 14.2a_1 + 214.6a_2 = 65.6$

$a_0 = 65.6 - 14.2\ (-40.264) - 214.6\ (1.036)$

$a_0 = 65.6 + 571.76 - 333.32 = 415.04$

$y = a_0 + a_1x + a_2x^2$

$y = 415.4 - 40.265x + 10.36x^2$ **Ans.**

Example 6.2. Fit a straight line to the above data. Fit curve of the form $y = a\ (b)^x$ by the method of least squares for the fallowing data.

x	0	1	2	3	4	5	6	7
y	10	21	35	59	92	200	400	610

(Supple. Nov. /Dec. 2005 Set 2)

(Supple.Feb.2010 Set.2)

Solution

x = X	y	Y = log y	XY	X^2
0	10	1	0	0
1	21	1.322	1.322	1
2	35	1.544	3.0588	4

3	
4	
5	
6	
7	
ΣX = 28	Σ

$A + 3.5B = 1.9$

$\underline{A + 5B\ \ = 2.2}$

$-\ 1.5B = -\ 0.3$

$B = 0.2$

$A + 3.5B = 1.9$

$A + 3.5(0.2543$

$A = 1.911 - 0.$

$A = 1.02084$

$A = \log a = 1.0$

$a = 10.49$

$B = \log b = 0.2$

$b = 1.79597$

$y = a\ b^x$

Example 6.3. Consic

3	59	1.7708	5.3124	9
4	92	1.963	7.852	16
5	200	2.3010	11.505	25
6	400	2.6020	15.612	36
7	610	2.7853	19.4991	49
$\Sigma X = 28$	$\Sigma y = 1427$	$\Sigma Y = 15.288$	$\Sigma XY = 64.1905$	$\Sigma X^2 = 140$

$$y = a\,b^x$$

Taking log

$$\log y = \log a + x \log b$$

$$Y = A + BX$$

Where

$$Y = \log y, A = \log a, B = \log b, X = x$$

Normal equations are

$$\sum y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

$$15.288 = 8A + 28B \ldots\ldots\ldots\ldots(1)$$

$$64.1905 = 28A + 140B \ldots\ldots\ldots(2)$$

Eq. (1) divided by 8

$$A + 3.5B = 1.911 \ldots\ldots\ldots(3)$$

Eq. (2) divided by 28

$$A + 5B = 2.2925 \ldots\ldots\ldots\ldots(4)$$

$A + 3.5B = 1.911$

$\underline{A + 5B \quad = 2.2925}$

$-1.5B = -0.3815$

$B = 0.2543$

$A + 3.5B = 1.911$

$A + 3.5(0.2543) = 1.911$

$A = 1.911 - 0.89016$

$A = 1.02084$

$A = \log a = 1.02084$

$a = 10.4915$

$B = \log b = 0.2543$

$b = 1.79597$

$y = a\,b^x$ $\qquad$ $y = (10.4915)(1.79597)^x$ $\qquad$ **Ans.**

Example 6.3. Consider the following data:

x	-4	-3	-2	-1	0	1	2	3	4
y	0.1	2.5	3.4	3.9	4.1	3.8	3.5	2.8	0.3

Find the correlation coefficient r.

(Supple.Feb.2007 Set 2)
(Nov. 2006 Set 4)

Solution

x	y	x^2	y^2	x, y
-4	0.1	16	0.01	-0.4
-3	2.5	9	6.25	-7.5
-2	3.4	4	11.56	-6.8
-1	3.9	1	15.21	-3.9
0	4.1	0	16.81	0
1	3.8	1	14.44	3.8
2	3.5	4	12.25	7.0
3	2.8	9	7.84	8.4
4	0.3	16	0.9	1.2
$\Sigma x = 0$	$\Sigma y = 24.4$	$\Sigma x^2 = 60$	$\Sigma y^2 = 84.46$	$\Sigma xy = 1.8$

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - \left(\sum_{i-1}^{n} x_i\right)^2 /n = 60 - (0)^2/9 = 60$$

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - \left(\sum_{i-1}^{n} y_i\right)^2 /n = 84.46 - (24.4)^2/9 = 84.46 - 66.15 = 18.30$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i-1}^{n} \hat{y}_i\right) /n = 1.8 - (0)(24.4)\ /9 = 1.8$$

$$r = \frac{1.8}{\sqrt{(60)(18.30)}} = \frac{1.8}{33.13} = 0.0543$$ **Ans.**

Example. 6.4. The measurements of humidity and the moisture content in a raw material are given in the following table. Fit a straight line of the form.

$y = ax + b$ Humidity (x)

42	35	50	43	48	62	31	36	44	39	55	48
12	8	14	9	1	16	7	9	12	10	13	1

Solution

x
42
35
50
43
48
62
31
36
44
39
55
48
$\Sigma x = 53$

Normal equatior

Eq. (1) divided

Eq. (2) divided

From (3) & (4)

$b + 46.02a =$

$b + 44.41$

1.61 a

$a = 0.2298$

$12b + 533a =$

$12b + 553(0.2$

$12b = 122 - 1$

(Nov. 2006 Set 4)

Solution

x	y	xy	x^2
42	12	504	1764
35	8	280	1225
50	14	700	2500
43	9	387	1849
48	1	48	2304
62	16	992	3844
31	7	217	961
36	9	324	1296
44	12	528	1936
39	10	390	1521
55	13	715	3025
48	11	528	2304
$\Sigma x = 533$	$\Sigma y = 122$	$\Sigma xy = 5613$	$\Sigma x^2 = 24529$

$$y = ax + b$$

Normal equations are

$$\Sigma y = nb + a\Sigma x$$

$$\sum xy = b\sum x + a\sum x^2$$

$$122 = 12b + 533a......(1)$$

$$5613 = 533b + 24529a.........(2)$$

Eq. (1) divided 12

$$b + 44.41a = 10.16.....(3)$$

Eq. (2) divided by 533

$$b + 46.02a = 10.53.....(4)$$

From (3) & (4)

$$b + 46.02a = 10.53$$

$$\underline{b + 44.41a = 10.16}$$

$$1.61\ a = 0.37$$

$$a = 0.2298$$

$$12b + 533a = 122$$

$$12b + 553(0.2298) = 122$$

$$12b = 122 - 122.04906$$

$$12b = 0.49068$$

$$b = 0.04089$$

$$y = ax + b$$

$$y = 0.2298x + 0.04089$$ **Ans.**

Example 6.5. The following data pertain to the cosmic ray doses measured at various attitudes

Altitude (feet x)	50	450	780	1200	4400	4800	5300
Dose (year y)	28	30	32	36	51	58	69

Fit a straight lines y = a + bx

(Supple.Nov./Dec.2005 Set 4)
(Supple. April/May 2005 Set 3)

Solution

x	y	xy	x^2
50	28	1400	2500
450	30	13500	202500
780	32	24960	608400
1200	36	43200	1440000
4400	51	224400	1936000
4800	58	278400	23040000
5300	69	365700	28090000
Σx = 16980	Σy = 304	Σxy = 951560	Σx^2 = 72743400

$$y = a + bx$$

Normal equations are

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

304 = 7a + 16980 b.......(1)

951560 = 16980 a + 72743400 b (2)

Eq (1) divided by 7

a + 2425.71 b = 43.428 ......(3)

Eq. (2) divided by 16980

a + 4284.06 b = 56.04 ....(4)

a + 2425.71 b = 43.428

1858.35 b = 1.29041

b = 0.000694

$7a + 16980b =$

$7a = 304 - 16$

$7a = 304 - 11$

$a = 41.74$

$y = a + bx$

$y = 41.79 + .($

Example 6.6. Fit a pa

To the followin

X	1	2
Y	1.28	1.53

Fit a parabola by the

Solution

x	y
1	1.28
2	1.53
3	1.03
4	0.81
5	0.74
6	0.65
7	0.87
8	0.81
9	1.10
10	1.03
Σx =55	Σy = 9.85

$y = a_0 + a_1x$

Normal equatic

$\sum y = a_0$

$\sum xy \doteq a$

$\sum x^2 y =$

$$7a + 16980b = 304$$

$$7a = 304 - 16980(0.000694)$$

$$7a = 304 - 11.7906$$

$$a = 41.74$$

$$y = a + bx$$

$$y = 41.79 + .000694x$$ **Ans.**

Example 6.6. Fit a parabola of the from $y = a_0 + a_1x + a_2x^2$

To the following data.

X	1	2	3	4	5	6	7	8	9	10
Y	1.28	1.53	1.03	0.81	0.74	0.65	0.87	0.81	1.10	1.03

Fit a parabola by the method of least squares and estimate y at x = 7.5

(Supple.Nov./Dec.2005 Set 3)
(Supple. April/May 2005 Set 1)
(Supple. Feb. 2007 Set 2)

Solution

x	y	xy	x^2	x^2y	x^3	x^4
1	1.28	1.28	1	1.28	1	1
2	1.53	3.06	4	6.12	8	16
3	1.03	3.09	9	9.27	27	81
4	0.81	3.24	16	12.96	64	256
5	0.74	3.7	25	18.5	125	625
6	0.65	3.9	36	23.4	216	1296
7	0.87	6.09	49	42.63	343	2401
8	0.81	6.48	64	51.84	512	4096
9	1.10	9.9	81	89.1	729	6561
10	1.03	10.3	100	103	1000	10000
$\Sigma x = 55$	$\Sigma y =$ 9.85	$\Sigma xy =$ 51.04	$\Sigma x^2 =$ 385	$\Sigma x^2y =$ 358.1	$\Sigma x^3 =$ 3025	$\Sigma x^4 =$ 25333

$$y = a_0 + a_1x + a_2x^2$$

Normal equations are

$$\sum y = a_0 n + a_1 x + a_2 x^2$$

$$\sum xy \doteq a_0 \sum x + a_1 \sum x^2 + a_2 \sum x^3$$

$$\sum x^2 y = a_0 \sum x^2 + a_1 \sum x^3 + a_2 \sum x^4$$

$$9.85 = 10a_0 + 55a_1 + 385a_2 \ldots\ldots\ldots\ldots(1)$$

$$51.04 = 55a_0 + 385a_1 + 3025a_2 \ldots\ldots\ldots(2)$$

$$358.1 = 385a_0 + 3025a_1 + 25333a_2 \ldots.(3)$$

Eq. (1) divided by 10

$$a_0 + 5.5a_1 + 38.5a_2 = 0.985 \ldots\ldots\ldots(4)$$

Eq. (2) divided by 55

$$a_0 + 7a_1 + 55a_2 = 0.928 \ldots\ldots\ldots(5)$$

Eq. (3) divided by 385

$$a_0 + 7.857a_1 + 65.8 = 0.9301 \ldots..(6)$$

From (4) and (5)

$$a_0 + 5.5a_1 + 38.5a_2 = 0.985$$

$$a_0 + 7a_1 + 55a_2 = 0.928$$

$$-1.5\,a_1 - 16.5\,a_2 = 0.057$$

$$1.5\,a_1 + 16.5\,a_2 = -0.057$$

Divided by 1.5

$$a_1 + 11\,a_2 = -0.038 \ldots\ldots\ldots\ldots (7)$$

from (5) and (6)

$$a_0 + 7a_1 + 55a_2 = 0.928$$

$$a_0 + 7.857a_1 + 65.8a_2 = 0.9301$$

$$-0.857\,a_1 - 10.8a_2 = -.0021$$

$$0.857\,a_1 + 10.8\,a_2 = .0021$$

Divided by 0.857

$$a_1 + 12.602a_2 = 0.00245 \ldots\ldots\ldots(8)$$

From (7) and (8)

$$a_1 + 11\,a_2 = -0.038$$

$$a_1 + 12.602a_2 = 0.00245$$

$$-1.602\,a_2 = -0.04045$$

$$a_2 = 0.0252$$

$$a_1 + 11a_2 = -0.038$$

$$a_1 + 11(0.0252) = -0.038$$

$$a_1 = -0.038 - 0.2772$$

$$a_1 = 0.3152$$

$$a_1 = 0.3152$$

$$a_0 + 7a_1 + 55a_2 = 0.928$$

$$a_0 + 7(0.3152) + 55(0.0252) = 0.928$$

$$a_0 = 0.928 - 2.2064 - 1.386$$

$$a_0 = -2.6644$$

$$y = -2.6644 + 0.3152x + 0.0252x^2$$ **Ans.**

Example 6.7. Compute the coefficient of correlation and the two lines of regression following data

x	14	16	17	18	19	20	21	22	23
y	84	78	70	75	66	67	62	58	60

Solution

x	
14	
16	
17	
18	
19	
20	
21	
22	
23	
Σx = 170	Σ

$S_{xx} = \sum_{i=1}^{n} x_i^2 - \left(\sum_{i-1}^{n} x_i\right.$

$S_{yy} = \sum_{i=1}^{n} y_i^2 - \left(\sum_{i-1}^{n} y\right.$

$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n}\right.$

$r = \frac{1}{\sqrt{(68.8}}$

$r = \frac{195.11}{204.4}$

The line of regress
$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x$

(Supple.Nov./Dec.2004 Set 3)

Solution

x	Y	x^2	y^2	xy
14	84	196	7056	1176
16	78	256	6084	1248
17	70	289	4900	1190
18	75	324	5625	1350
19	66	361	4356	1254
20	67	400	4489	1340
21	62	441	3844	1302
22	58	484	3364	1276
23	60	529	3600	1380
$\Sigma x = 170$	$\Sigma y = 620$	$\Sigma x^2 = 3280$	$\Sigma y^2 = 42318$	$\Sigma xy = 11516$

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 3280 - \frac{(170)^2}{9} = 3280 - 3211.11 = 68.888$$

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/ n = 43318 - \frac{(620)^2}{9} = 43318 - 42711.11 = 606.888$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\Big/ n = 11516 - \frac{(170)(620)}{9} = 11516 - 11711.11 = 195.111$$

$$r = \frac{195.1111}{\sqrt{(68.888)(606.888)}} = \frac{195.1111}{\sqrt{41807.30}}$$

$$r = \frac{195.11111}{204.468} = 0.9542$$

Ans.

The line of regression of y on x	The line of regression of x on y
$y - \overline{y} = r\frac{\sigma_y}{\sigma_x}(x - \overline{x})$	$x - \overline{x} = r\frac{\sigma_x}{\sigma_y}(y - \overline{y})$

$\sigma_x^2 = \frac{1}{n}\sum x^2 - \bar{x}^2$	$\sigma_y^2 = \frac{1}{n}\sum y^2 - \bar{y}^2$
$= \frac{1}{9}3280 - \left(\frac{170}{9}\right)^2$	$= \frac{1}{9}43318 - \left(\frac{620}{9}\right)^2$
$= 364.44 - 356.79$	$\sigma_y^2 = 4813.11 - 4745.67$
$\sigma_x^2 = 7.6498$	$\sigma_y^2 = 67.930$
$\sigma_x = 2.7658$	$\sigma_y = 8.2115$

$$y - 68.88 = \frac{0.9542 \times 8.2115}{2.7658}(x - 18.888)$$

$$y - 68.88 = 2.83296(x - 18.888)$$

$$y - 68.88 = 2.83296x \times 53.5113$$

$$y = 2.83296x - 53.5113 + 68.88$$

$$y = 2.83296x + 15.368$$

Ans.

$$x - 18.888 = \frac{0.9542 \times 2.7658}{8.2115}(y - 68.88)$$

$$x - 18.888 = 0.32139(y - 68.88)$$

$$x - 18.888 = 0.32139y - (22.1376)$$

$$x - 0.32139y = 18.888 - 22.1376$$

$$x - 0.32139y = -3.2496$$

$$x = 0.32139y - 3.2496$$

Ans.

Example 6.8. Fit a straight line y = a + bx for the following data

x	1	2	3	4	5	6
y	14	33	40	63	76	85

(November 2006)

Solution

x	y	xy	x^2
1	14	14	1

2	33	66	4
3	40	120	9
4	63	252	16
5	76	380	25
6	85	510	36
$\Sigma x = 21$	$\Sigma y = 311$	$\Sigma xy = 1342$	$\Sigma x^2 = 91$

Given curve is $y = a + bx$

Normal equations are

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

$$311 = 6a + 21b..........(1)$$

$$1342 = 21a + 91b......(2)$$

Eq. (1) Divided by 6

$$a + 3.5b = 51.83$$

Eq. (2) Divided by 21

$$a + 4.33b = 63.90$$

$$\underline{a + 3.5b = 51.83}$$

$$0.83\ b = 12.07$$

$$b = 14.54$$

$$a + 3.5b = 51.83$$

$$a = 51.83 - 3.5(14.54)$$

$$a = 51.83 - 50.89$$

$$a = 0.9324$$

$$y = a + bx$$

$$y = 0.9324 + 14.54x$$

Ans.

Example 6.9. Fit a curve of the $y = ax^b$ for the following data

x	1	2	3	4	5

y	0.5	0.2	4.5	8	12.5

(Supple Feb 2007)
(November 2006 Set 3)

Solution

x	y	X = log x	Y = log y	XY	X^2
1	0.5	0	-0.3010	0	0
2	0.2	0.3010	-0.6989	0.2103	0.0960
3	4.5	0.47712	0.6532	0.3116	0.22764
4	8	0.6020	0.9030	0.5436	0.3624
5	12.5	0.6989	1.0969	0.7666	0.48846
		ΣX = 2.07902	ΣY= 1.6532	ΣXY= 1.8321	$\Sigma X^2 = 1.16910$

Given curve is $y = ax^b$

Taking logarithms Log y = log a + b log x

Y = A + BX

Where Y = log y, A = log a, B = b, X = log x

Normal equations are

$$\sum Y = nA + B\sum X ....(1)$$

$$\sum XY = A\sum X \times B\sum X^2 ....(2)$$

1.6532 = 5A + 2.07902 B..........(1)

1.8321 = 2.07902A + 1.16910 B ....(2)

Eq. (1) Divided by 5

A + 0.415804 B = 0.33064 ....(3)

Eq. (2) Divided by 2.07902

A + 0.5623B = 0.8812 ....(4)

A + 0.5623B = 0.8812

A + 0.415804 B = 0.33064

0.1464B = 0.55056

B = 3.7606 = b

A + 0.5623B = 0.8812

A = 0.8812 – (0.5623) (3.7606)

A = 0.8812 – 2.1145

A = 1.2333

A = log a = 1.2333

a = 17.11

$y = ax^b$

$y = 17.1$

Example 6.10.

Solution

	Y
1	2.98
2	4.26
3	5.21
4	6.10
5	6.80
6	7.50

Normal equa

$\sum$

$\sum$

Eq. (1) Di

Eq. (2) Di

a = 17.115

$$y = ax^b$$

$$y = 17.115x^{3.7606}$$

Ans.

Example 6.10. Predict y at x = 3.75 by fitting a power curve to the given data.

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.10	6.80	7.50

(Reg. April/May 2004 Set 1)

Solution

We know that power curve is $y = ax^b$

Taking logarithms

Log y = log a + b log x

Y = A + B X

Where Y = log y, A = log a, B = b, X = log x

	Y	Y = log y	X = log x	X^2	XY
1	2.98	0.47421	0	0	0
2	4.26	0.62940	0.3010	0.090601	0.18944
3	5.21	0.71683	0.47712	0.227643	0.342013
4	6.10	0.78532	0.60205	0.36246	0.472801
5	6.80	0.83250	0.69897	0.48855	0.581892
6	7.50	0.87506	0.77815	0.605517	0.680927
		ΣY = 4.31332	ΣX = 2.85729	ΣX^2 = 1.77477	ΣXY = 2.267073

Normal equations are

$$\sum Y = nA + B\sum X \ldots\ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \ldots\ldots(2)$$

4.31332 = 6A + 2.85729 B ....(1)

2.267073 = 2.85729 A + 1.77477B.........(2)

Eq. (1) Divided by 6

A + 0.476215 B = 0.718886...... (3)

Eq. (2) Divided by 2.85729

A + 0.621137B = 0.79395....... (4)

$$A + 0.476215\,B = 0.718886$$

$$\underline{A + \ 0.621137B = 0.79395}$$

$$-\,0.144922\,B = -\,0.07459$$

$$B = 0.51469 = b$$

$$A + 0.476215B = 0.71886$$

$$A = 0.71886 - 0.476215(0.51469)$$

$$A = 0.71886 - 0.245103$$

$$A = 0.473756$$

$$A = \log a = 0.473756$$

$$a = 2.97684$$

$$y = ax^b = 2.97684x^{0.51469}$$ **Ans.**

Example 6.11. Calculate the correlation coefficient r for the following data

x	63	50	55	65	55	70	64	70	58	68	52	60
y	87	74	76	90	85	87	92	98	82	91	77	78

Solution

x	y	x^2	y^2	xy
63	87	3969	7569	5481
50	74	2500	5476	3700
55	76	3025	5776	4180
65	90	4225	8100	5850
55	85	3025	7225	4675
70	87	4900	7569	6090
64	92	4096	8464	5888
70	98	4900	9604	6860
58	82	3364	6724	4756
68	91	4624	8281	6188
52	77	2704	5929	4004
60	78	3600	6084	4680
$\Sigma x=730$	$\Sigma y=937$	$\Sigma x^2 = 44932$	$\Sigma y^2 = 86801$	$\Sigma xy = 62352$

$S_{xx} = \sum_{i=1}^{n} x_i^2 - \Big(\sum$

$S_{yy} = \sum_{i=1}^{n} y_i^2 - \Big($

$S_{xy} = \sum_{i=1}^{n} x_i y_i - $

$= 62353 - (7$

$r = \dfrac{S_{xy}}{\sqrt{S_{xx} S}}$

$r = \dfrac{5}{\sqrt{523.6}}$

Example 6.12. Fit

Solution

x = X	
1	
2	1
3	1
4	2
$\Sigma X=10$	Σ

$y = Ae^{Bx}$

$\log y = \log$

$\log y = \log$

$Y = A^* + B^*$

Where Y = lo

Normal equati

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 44932 - (730)^2 \;/12 = 44932 - 44408.33 = 523.666$$

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/_{n} = 86801 - (937)^2/12 = 86801 - 73164.08 = 13636.92$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n$$

$$= 62353 - (730)(937)/12 = 62352 - 57000.83 = 5351.16$$

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

$$r = \frac{5351.16}{\sqrt{523.66 \times 13636.92}} = \frac{5351.16}{2672.285} = 2.0024$$

Ans.

Example 6.12. Fit an exponential curve of the $y = Ae^{Bx}$ for the following data

x	1	2	3	4
y	7	11	17	27

(Reg. April/May 2004 Set No.2, Set No.3)

Solution

x = X	y	Y = log y	X^2	XY
1	7	0.8450	1	0.8450
2	11	1.0413	4	2.0826
3	17	1.2304	9	3.6912
4	27	1.4313	16	5.7252
ΣX=10	Σy =	Σy=4.5480	$\Sigma X^2 = 30$	ΣXY = 12.344

$$y = Ae^{Bx}$$

$$\log y = \log A + Bx \log_{10} e$$

$$\log y = \log A + B \log_{10} e.x$$

$$Y = A^* + B^* X$$

Where Y = log y, $A^* = \log A$, $B^* = B \log_{10} e$, X = x

Normal equations are

$$\sum Y = nA^* + B^* \sum X........(1)$$

$$\sum XY = A^* \sum X \times B^* \sum X^2 ....(2)$$

$4.5480 = 4A^* + 10B^*$.......... (1)

$12.344 = 10A^* + 30B^*$...... (2)

Eq. (1) Divided by 4

$A^* + 2.5\ B^* = 1.137$

Eq. (2) Divided by 10

$A^* + \ \ 3\ B^* = 1.2344$

$A^* + 2.5\ B^* = 1.137$

$0.5\ B^* = 0.974$

$B^* = 0.1948$

$A^* + 3B^* = 1.2344$

$A^* \times 3\ (0.1948) = 1.2344$

$A^* = 1.2344 - 0.5844$

$A^* = 0.65$

$A^* = \log A = 0.65$

$A = 4.466$

$B^* = B\log_{10} e = 0.1948$

$B(0..4342) = 0.1948$

$B = 0.4486$

$y = Ae^{Bx}$

$y = 4.466\ e^{0.4486x}$ **Ans.**

Example 6.13. The marks obtained by 10 students in mathematics and statistics are given below. Find the coefficient of correlation between the two subjects and the two lines of regression.

Marks in maths	75	30	60	80	53	35	15	40	38	48
Marks in statistics	85	45	54	91	58	63	35	43	45	44

(Supple. Feb.2007 Set 1)
(Nov. 2006 Set 1)

Solution

x
75
30
60
80
53
35
15
40
38
48
Σx=474

$r = \frac{}{\sqrt{S}}$

$S_{xx} = \sum_{i=1}^{n} x_i^2 - ($

$S_{yy} = \sum_{i=1}^{n} y_i^2 - ($

$S_{xy} = \sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n}$

$r = \frac{}{\sqrt{(36}}$

$= \frac{2}{\sqrt{11}}$

$= \frac{290}{3380}$

$r = 0.8593$

1. The line of re
2.

statistics are
and the two

38	48
45	44

.2007 Set 1)
. 2006 Set 1)

Solution

x	y	x2	y2	xy
75	85	5625	7225	6375
30	45	900	2025	1350
60	54	3600	2916	3240
80	91	6400	8281	7280
53	58	2809	3364	3074
35	63	1225	3969	2205
15	35	225	1225	525
40	43	1600	1849	1720
38	45	1444	2025	1710
48	44	2304	1936	2112
Σx=474	Σy=563	$\Sigma x^2 = 26132$	$\Sigma y^2 = 34815$	Σxy = 29591

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$S_{xx} = \sum_{i=1}^{n} x_i^{\,2} - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 26132 - \frac{(474)^2}{10} = 26132 - 22467.6 = 3664.4$$

$$S_{yy} = \sum_{i=1}^{n} y_i^{\,2} - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/ n = 34815 - -\frac{(563)^2}{10} = 34815 - 31696.9 = 31181$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = 29591 - \frac{(474)(563)}{10} = 29591 - 26686.2 = 2904.8$$

$$r = \frac{2904.8}{\sqrt{(3664.4)\times(3118.1)}}$$

$$= \frac{2904.8}{\sqrt{11425965.6}}$$

$$= \frac{2904.8}{3380.23}$$

$r = 0.8593$ **Ans.**

1. The line of regression of Y on X
2.

The line of regression of y on x	The line of regression of x on y
$y-\bar{y}=r\dfrac{\sigma_y}{\sigma_x}(x-\bar{x})$	$x-\bar{x}=r\dfrac{\sigma_x}{\sigma_y}(y-\bar{y})$
$\sigma_x^2=\dfrac{1}{n}\sum x^2-\bar{x}^2$	$\sigma_y^2=\dfrac{1}{n}\sum y^2-\bar{y}^2$
$=\dfrac{1}{10}\,26132-(47.4)^2$	$=\dfrac{1}{10}34815-(56.3)^2$
$\sigma_x^2=2613.2-2246.76$	$\sigma_y^2=3481.5-3169.69$
$\sigma_x^2=366.44$	$\sigma_y^2=311.81$
$\sigma_x=19.142$	$\sigma_y=17.65$

$$y-56.3=0.859\times\frac{17.65}{19.142}(x-47.4) \qquad x-47.4=0.8593\times\frac{19.142}{17.65}(y-56.3)$$

$$y-56.3=0.792(x-47.4) \qquad x-47.4=0.9319(y=56.3)$$

$$y-56.3=0.792x-37.54 \qquad x-47.4=0.9319y-52.468$$

$$y-0.792x=-37.54+56.3 \qquad x-0.9319y=-52.468+47.4$$

$$y-0.792x=18.76 \textbf{ Ans.} \qquad x-0.9919y=-5.068 \textbf{ Ans.}$$

Example 6.14. Determine the least square regression line of

a. y on x
b. x on y
c. Find r using the regression co effects
d. Find y (8)
e. Find x (16)

x	12	10	14	11	12	9
y	18	17	23	19	20	15

(Reg. April/May 2004 Set 2)

Solution

X	y	x^2	y^2	xy
12	18	144	324	216
10	17	100	289	170
14	23	196	529	322
11	19	121	361	209
12	20	144	400	240
9	15	81	225	135
Σx=68	Σy=112	$\Sigma x^2 = 786$	$\Sigma y^2 = 2128$	Σxy = 1292

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$S_{xx} = \sum_{i=1}^{n} x_i^{\,2} - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 786 - (68)^2 \,/_6 = 786 - 770.66 = +15.333$$

$$S_{yy} = \sum_{i=1}^{n} y_i^{\,2} - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/_n = 2128 - (112)^2 /_6 = 2128 - 2090.66 = 37.333$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i^{\,2} - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = 1292 - (68)(112)/_6 = 1292 - 1269.37 = 22.665$$

$$r = \frac{22.666}{\sqrt{(15.333)(37.333)}} = \frac{22.666}{\sqrt{572.426}} = \frac{22.666}{23.925}$$

$$r = 0.9473$$

Ans.

a. The line of regression of y on x	b. The line of regression of x on y
$y - \bar{y} = r\dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$	$x - \bar{x} = r\dfrac{\sigma_x}{\sigma_y}(y - \bar{y})$
$\sigma_x^2 = \dfrac{1}{n}\sum x^2 - \bar{x}^2$	$\sigma_y^2 = \dfrac{1}{n}\sum y^2 - \bar{y}^2$
$= \dfrac{786}{6} - \left(\dfrac{68}{6}\right)^2$	$\dfrac{2128}{6} - \left(\dfrac{112}{6}\right)^2$
$\sigma_x^2 = 131 - 128.44$	$\sigma_y^2 = 354.666 - 348.444$
$\sigma_x^2 = 2.555$	$\sigma_y^2 = 6.2215$
$\sigma_x = 1.598$	$\sigma_y = 2.49429$

$$-18.66 = (0.9473)\times\frac{(2.4929)}{1.598}(x - 11.33)$$

$$-18.66 = 1.4777(x - 11.33)$$

$$-18.66 = 1.4777x - 16.743$$

$$= 1.4777x - 16.743 + 18.66$$

$$= 1.4777x + 1.917 \quad \textbf{Ans.}$$

$$x - 11.33 = (0.9473)\times\frac{(1.598)}{(2.49429)}(y - 18.66)$$

$$x - 11.33 = 0.60900(y - 18.66)$$

$$x - 11.33 = 0.606900y - 11.324$$

$$x = 0.6069y \quad \textbf{Ans.}$$

d. y (8)

$$y = 1.4777(8) + 1.917$$

y = 13.73 Ans.

e. x (16)

x = 0.6069 (16)

x = 9.7104 **Ans.**

Example 6.15. Estimate y at x = 5 by fitting a least square curve of the form $y = \dfrac{b}{x(x-a)}$ to the following data

x	3.6	4.8	6.0	7.2	8.4	9.6	10.8
y	0.83	0.31	0.17	0.10	0.07	0.05	0.04

(Reg. April/May 2004 Set 4)

Solution

x =X	Y	Y=1/y	X^2	X^3	XY
3.6	0.83	1.204	12.96	46.656	4.3344
4.8	0.31	3.225	23.04	110.59	15.48
6.0	0.17	5882	36.0	216	35.292
7.2	0.10	10	5184	373.248	72.00
8.4	0.07	14.2587	70.56	592.704	119.999
9.6	0.05	20	92.16	884.736	192
10.8	0.04	25	116.64	1259.71	270
ΣX=50.4	Σy=	ΣY= 79.59	$\Sigma X^2 = 403.2$	ΣX^3=3483.64	ΣXY = 709.1054

Given equation is $y = \dfrac{b}{x(x-a)}$

n = 7

$$\text{Or } \frac{1}{y} = \frac{x(x-a)}{b}$$

$$\frac{1}{y} = \frac{x^2}{b} - \frac{ax}{b}$$

$$Y = \frac{1}{b}X^2 - \frac{a}{b}X$$

Or

Then the nor

A + 8E

A + 8.63

- 0.639

A + 8B =

A + 8(0.

A = 1.57

A = - 0.6

$B = \frac{1}{b} =$

$b = \dfrac{1}{0.2810}$

b = 3.55

$A = -\dfrac{a}{b}$

$$Y = -\frac{a}{b}X + \frac{1}{b}X^2$$

$$Y = AX + BX^2$$

$$\text{Where } Y = \frac{1}{y}, X = x,\ A = -\frac{a}{b},\ B = \frac{1}{b}$$

$$y = AX + BX^2$$

Then the normal equations are

$$\Sigma Y = A\sum X + B\Sigma X^2 \ldots\ldots(1)$$

$$\Sigma XY = A\Sigma X^2 + B\Sigma X^3 \ldots\ldots(2)$$

$$79.58 = A(50.4) + B(403.2)\ldots..(1)$$

$$709.10 = A(403.2) + B(3483.64)\ldots..(2)$$

Eq. (1) Divided by 50.4

A + 8B = 1.579……….(3)

Eq. (2) Divided by 403.2

A + 8.639 B = 1.7586 ……..(4)

A + 8B = 1.579

A + 8.639 B = 1.7586

- 0.639 B = - 0.1796

B = 0.2810

A + 8B = 1.579

A + 8(0.2810) = 1.579

A = 1.579 – 2.248

A = - 0.669

$$B = \frac{1}{b} = 0.2810$$

$$b = \frac{1}{0.2810}$$

b = 3.5587

$$A = -\frac{a}{b} = -0.669$$

$$a = b(0.669)$$

$$= (3.5587)(0.669)$$

$$a = 2.3807$$

$$y = \frac{b}{x(x-a)} = \frac{3.5587}{x(x-2.3807)}$$ **Ans.**

Example 6.16. Fit the curve $y = ae^{bx}$ for the following data

x	1	5	7	9	12
y	10	15	12	15	21

(Reg. April/ May 2005 Set 2)

Solution

x =X	y	Y= $\log_{10} y$	XY	X^2
1	10	1	1	1
5	15	1.1760	5.88	25
7	12	1.0791	7.5537	49
9	15	1.1760	10.584	81
12	0.0217	1.3222	15.8664	144
ΣX=34	Σy= 73	ΣY= 5.7533	ΣXY=40.8841	ΣX^3=300

Given curve is $y = ae^{bx}$

Taking logarithms of both sides

$$\log_{10} y = \log_{10} a + b\,x \log_{10} e$$

$$Y = A + BX$$

Where $Y = \log_{10} y$, $A = \log_{10} a$, $X = x$, $B = b \log_{10} e$

Normal equations are

$$\sum Y = nA + B\sum X \text{.......}(1)$$

$$\sum XY = A\sum X + B\sum X^2 \text{........}(2)$$

5.7533 = 5A + 34 B .... (1)

40.8841 = 34A + 300 B........ (2)

Eq. (1) Divided by 5

1.15066 = A + 6.8 B........ (3)

Eq. (2) Divided by 34

1.2024 = A + 8.8235 B....... (4)

A + 6.

A + 8.8

- 2.

A + 6.8

A + 6.8

A = 1.1

A = 0.9

A = log

a = 9.47

B = b lo

$b = \frac{0.0}{\log}$

$b = 0.0$

$y = ae^{b}$

$y = 9.4$

Example 6.17.

The followin

kind of forged allo

metal. Fit a least seq

No. of twists	y	41	4
% of elements of A	x	1	2
% of elements of B	x_2	5	5

Ans.

May 2005 Set 2)

X^2
1
25
49
81
144
ΣX^3=300

$$A + 6.8\,B = 1.15066$$
$$A + 8.8235\,B = 1.2024$$

$$-2.0235B = -0.05174$$
$$B = 0.0255$$

$$A + 6.8\,B = 1.15066$$

$$A + 6.8\,(0.0255) = 1.15066$$

$$A = 1.15066 - 0.173872$$

$$A = 0.976788$$

$$A = \log_{10} a = 0.976788$$

$$a = 9.4795$$

$$B = b \log_{10} e = 0.0255$$

$$b = \frac{0.0255}{\log_{10} e} = \frac{0.0255}{0.4342}$$

$$b = 0.05872$$

$$y = ae^{bx}$$

$$y = 9.4795e^{(0.05872)x}$$

Ans.

Example 6.17.

The following are the data on the number of twists required to break a certain kind of forged alloy bar and the percentage of two alloying elements present in the metal. Fit a least sequence regression line of on x_1 and x_2.

(Reg. April/ May 2005)

No. of twists	y	41	49	69	65	40	50	58	57	31	36	44	57	19	31	33	43
% of elements of A	x	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
% of elements of B	x_2	5	5	5	5	10	10	10	10	15	15	15	15	20	20	20	20

Solution

No. of twist y	% of element of A x_1	% element of B x_2	x_1y	x_1^2	x_1x_2	x_2y	x_2^2
41	1	5	41	1	5	205	25
49	2	5	98	4	10	245	25
69	3	5	207	9	15	345	25
65	4	5	260	16	20	325	25
40	1	10	40	1	10	400	100
50	2	10	100	4	20	500	100
58	3	10	174	9	30	580	100
57	4	10	228	16	40	570	100
31	1	15	31	1	15	465	225
36	2	15	72	4	30	540	225
44	3	15	132	9	45	660	225
57	4	15	228	16	60	855	225
19	1	20	19	1	20	380	400
31	2	20	62	4	40	620	400
33	3	20	99	9	60	660	400
43	4	20	172	16	80	860	400
$\Sigma y = 723$	$\Sigma x_1 = 40$	$\Sigma x_2 = 200$	$\Sigma x_1y = 1963$	$\Sigma x_1^2 = 120$	$\Sigma x_1x_2 = 500$	Σx_2y 8210	$\Sigma x_2^2 = 3000$

Normal equations are

$$\Sigma y = nb_0 + b_1\Sigma x_1 + b_2\Sigma x_2$$

$$\Sigma x_1 y = b_0\Sigma x_1 + b_1\Sigma x_1^2 + b_2\Sigma x_1x_2$$

$$\Sigma x_2 y = b_0\Sigma x_2 + b_1\Sigma x_1x_2 + b_2\Sigma x_2^2$$

$$723 = 16b_0 + 40b_1 + 200b_2$$

$$1963 = 40b_0 + 120b_1 + 500b_2$$

$$8210 = 200b_0 + 500b_1 + 3000b_2$$

The unique solution of this system of equations is

$b_0 = 46.4,\ b_1 = 7.78,\ b_2 = -1.65$

$y = 46.4 + 7.78x_1 - 1.65x_2$ **Ans.**

Example 6.18. Fi
time and the amount

Varnish additiv (x in grams)
Drying time (y in hours)

Solution

Varnish additive(x in grams)	Dryi (y in
0	1:
1	10
2	1(
3	8
4	7
5	8
6	7
7	8
8	9
$\Sigma x = 36$	Σy= 8(

Normal equatic

$\sum y = na$

$\sum xy = a$

$\sum x^2 y$:

Example 6.18. Fit a quadratic function to the following data which gives the drying time and the amount of additive that is intended to reduce the drying time.

Varnish additive (x in grams)	0	1	2	3	4	5	6	7	8
Drying time (y in hours)	12.0	10.5	10.0	8.0	7.0	8.0	7.5	8.5	9.0

(November 2006)
(Supple. Feb. 2010 Set 1, 2, 3, 4)

Solution

Varnish additive(x in grams)	Drying time (y in hours)	x^2	x^3	x^4	x y	x^2y
0	12.0	0	0	0	0	0
1	10.05	1	1	1	10.5	10.5
2	10.0	4	8	16	20.0	40.0
3	8.0	9	27	81	24.0	72.0
4	7.0	16	64	256	28.0	112.0
5	8.0	25	125	625	40.0	200.0
6	7.5	36	216	1296	45	270.0
7	8.5	49	343	2401	59.5	416.0
8	9.0	64	512	4096	72.0	576.0
$\Sigma x = 36$	$\Sigma y = 80.5$	$\Sigma x^2 = 204$	$\Sigma x^3 = 1296$	$\Sigma x^4 = 8772$	$\Sigma xy = 299$	$\Sigma x^2y = 1697$

$$y = a + bx + cx^2$$

Normal equations are

$$\sum y = na + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + cx^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

$80.5 = 9a + 36b + 204c$ .... (1)

$299.0 = 36a + 204b + 1296c$ ...... (2)

$1697.0 = 204a + 1296b + 8772c$ ...... (3)

Eq. (1) Divided by 9

$$a + 4b + 22.66c = 8.944 ....(4)$$

Eq. (2) Divided by 36

$$a + 5.666b + 36c = 8.305 ......(5)$$

Eq. (3) Divided by 204

$$a + 6.352b + 43c = 8.318 ....(6)$$

$$\begin{array}{r} a + 4b + 22.66c = 8.944 \\ \underline{a + 5.666b + 36c = 8.305} \\ -1.666b - 13.34c = 0.639 \end{array} \qquad \begin{array}{r} a + 5.666b + 36c = 8.305 \\ \underline{a + 6.352b + 43c = 8.318} \\ -0.686b - 7c = -0.013 \end{array}$$

$1.666b + 13.34c = -0.639$ .... (7)

$0.686b + 7c = 0.013$ .... (8)

$$\begin{array}{r} 11.662b + 93.38c = -4.473 \\ \underline{9.15124b + 93.38c = 0.17342} \\ 2.51076b \qquad = -4.64642 \end{array}$$

$$b = -1.8506$$

$0.686b + 7c = 0.013$

$0.686(-1.8506) + 7c = 0.013$

$-1.2213 + 7c = 0.013$

$7c = 0.013 + 1.2213$

$7c = 1.23439$

$c = 0.1763$

$9a + 36b + 204c = 80.5$

$9a + 36(-1.8506) + 204(0.1763) = 80.5$

$9a = 80.5 - 35.973 + 66.6216$

$9a = 111.148$ $\qquad a = 12.34$

$y = 12.34 - 1.8506x + 0.1763x^2$ **Ans.**

Example 6.19. 10 observations on price x and supply y the following data was obtained

$$\Sigma x = 130,\ \Sigma y = 220,\ \Sigma x^2 = 2288,\ \Sigma y^2 = 5506 \text{ and } \Sigma xy = 3467$$

Find

a. Coefficient of correlation
b. The line of regression of y or x
c. The standard error of estimate

(Nov. 2006 Set 3)
(Supple. Nov. /Dec. 2005 Set 2)

Solution

a. Coefficient of correlation

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$S_{xx} = \sum_{i=1}^{n} x_i^{\,2} - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 2288 - (130)^2 \ /_{10} = 2288 - 1690 = 598$$

$$S_{yy} = \sum_{i=1}^{n} y_i^{\,2} - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/_{n} = 5506 - (220)^2/_{10} = 5506 - 4840 = 666$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = 3467 - -(130)(220)/_{10} = 3467 - 2860 = 607$$

$$r = \frac{607}{\sqrt{(598)(666)}} = \frac{607}{\sqrt{398268}} = \frac{607}{631.08} = 0.9618$$

Ans.

b. The line of regression of y on x

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$\sigma_x^2 = \frac{1}{n}\Sigma x^2 - \bar{x}^2 = \frac{2288}{10} - (13)^2 = 228.8 - 169 = 59.8$$

$$\sigma_x = 7.73$$

$$\sigma_y^2 = \frac{1}{n}\Sigma y^2 - \bar{y}^2 = \frac{5506}{10} - (22)^2 = 550.6 - 484 = 66.6$$

$$\sigma_y = 8.16$$

$$y - 22 = 0.9618 \times \frac{8.16}{7.73}(x - 13)$$

$$y - 22 = 1.0154(x - 13)$$

$$y - 22 = 1.0154x - 13.200$$

$$y - 1.0154x = 8.7996$$

Ans.

c. The standard error of estimate

$$Se^2 = \frac{S_{yy} - {S_{xy}}^2 / S_{xx}}{n-2} = \frac{666 - (607)^2/598}{10-2} = \frac{666 - 616.135}{8}$$

$$= \frac{49.86}{8} = 6.23$$

$$S_e^2 = 6.23$$

$$S_e = 2.495 \textbf{ Ans.}$$

Example 6.20. Fit a second degree polynomial to the following data, taking x as independent variable

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	15

(Supple. Nov. /Dec. 2004 Set 2)

Solution

Equation of the second degree polynomial

$$y = a + bx + cx^2$$

x	y	x^2	x^3	x^4	xy	x^2y
1	2	1	1	1	2	2
2	6	4	8	16	12	24
3	7	9	27	81	21	63
4	8	16	64	256	32	128
5	10	25	125	625	50	250
6	11	36	216	1296	66	396
7	11	49	343	2401	77	539
8	10	64	512	4096	80	640
9	15	81	729	6561	135	1215
$\Sigma x = 45$	$\Sigma y = 80$	$\Sigma x^2 = 285$	$\Sigma x^3 = 2025$	$\Sigma x^4 = 15333$	$\Sigma xy = 475$	$\Sigma x^2y = 3257$

Normal Equations are

$$\sum y = na + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + c\mathbf{x}^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

taking x as

. 2004 Set 2)

x^2y
2
24
63
128
250
396
539
640
1215
$\Sigma x^2y = 3257$

80 = 9a + 45b + 285c .... (1)
475 = 45a + 285b + 2025c .... (2)
3257 = 285a + 2025b + 15333c .... (3)

400 = 45a + 225b + 1425 c
475 = 45a + 285b + 2025c

- 75 = - 60b – 600c

60b + 600c = 75

22800 = 2565a + 12825b + 81225
29313 = 2565a + 18225 b + 1379970

- 6513 = - 5400b – 567720

5400b + 56772 c = 6513

Eq. (1) Divided by 6

$$b_0 + 7b_1 + 50.33b_2 = 8..........(4)$$

Eq. (2) Divided by 42

$$b_0 + 10.33b_1 + 67.23b_2 = 5.61....(5)$$

Eq. (3) Divided by 302

$$b_0 + 9.35b_1 + 63.11b_2 = 6.112......(6)$$

From (4) and (5)

$$b_0 + 7b_1 + 50.33b_2 = 8$$

$$b_0 + 10.33b_1 + 67.23b_2 = 5.61$$

$- 3.33\, b_1 - 16.9b_2 = 2.39$

From (5) and (6)

$$b_0 + 10.33b_1 + 67.23b_2 = 5.61$$

$$b_0 + 9.35b_1 + 63.11b_2 = 6.112$$

$0.98b_1 + 4.12b_2 = -0.502$

$3.33\, b_1 + 16.9b_2 = -2.39$ ...... (7)

$0.98b_1 + 4.12b_2 = -0.502$ .... (8)

Eq. (7) Multiplied by 0.98

$3.2634b_1 + 16.562b_2 = 2.3422$

$-3.2634b_1 + 13.71966b_2 = - 1.67166$

$2.8424b_2 = -0.67054$

$$b_2 = \frac{-0.67054}{2.8424}$$

$$b_2 = -0.2359$$

From Eq. (7)

$$3.33b_1 + 16.9\,b_2 = -2.39$$

$$3.33\,b_1 + 16.9(-0.2359) = -\,2.36$$

$$3.33\,b_1 - 3.9868 = -2.39$$

$$3.33\,b_1 = -\,2.39 + 3.9868$$

$$3.33\,b_1 = 1.5968$$

$$b_1 = 0.4795$$

$$b_0 + 7b_1 + 50.33b_2 = 8$$

$$b_0 = 8 - 7 \times (0.4795) - 50.33(-0.2359)$$

$$b_0 = 8 - 3.3565 + 11.8728$$

$$b_0 = 16.5163$$

$$x_1 = b_0 + b_1x_2 + b_2x_3$$

$$= 16.5163 + 0.4795\,x_2 + (-0.2359)x_3$$ **Ans.**

Example 6.21. Find the least squares regression equation of X_1 and X_2 and X_3 from the following data.

X_1	3	5	6	8	12	14
X_2	16	10	7	4	3	2
X_3	90	72	54	42	30	14

(November 2006 Set 2)

Solution

X_1	X_2	X_3	X_1X_2	X_2^2	X_2X_3	X_1X_3	X_3^2
3	16	90	48	256	1440	270	8100
5	10	72	50	100	720	360	5184
6	7	54	42	49	378	324	2916
8	4	42	32	16	168	336	1764
12	3	30	36	9	90	360	900
14	2	14	28	4	28	196	196
$\Sigma X_1 =$ 48	ΣX_2 =42	ΣX_3 =302	$\Sigma X_1X_2 =$ 236	$\Sigma X_2^2 =$ 434	$\Sigma X_2X_3 =$ 2824	ΣX_1X_2 = 1846	$\sum X_3^2$ =19060

Normal equations for multiple regression with r = 2

$$\sum y = nb_0 + b_1 \sum X_1 + b_2 \sum X_2$$

$$\Sigma X_1 Y = b_0 \Sigma X_1 + b \Sigma X_1^2 + b_2 \Sigma X_1 X_2$$

$$\Sigma X_2 Y = b_0 \Sigma X_2 + b_1 \Sigma X_1 X_2 + b_2 \Sigma X_2^2$$

Now let

$$X_1 = b_0 + b_1 X_2 + b_2 X_3$$

normal equations are

$$\Sigma X_1 = nb_0 + b_1 \Sigma X_2 + b_2 \Sigma X_3 \ldots (1)$$

$$\Sigma X_2 X_1 = b_0 \Sigma X_2 + b_1 \Sigma X_2^2 + b_2 \Sigma X_2 X_3 \ldots\ldots (2)$$

$$\Sigma X_3 X_1 = b_0 \Sigma X_3 + b_1 \Sigma X_2 X_3 + b_2 \Sigma X_3^2 \ldots\ldots\ldots (3)$$

$$48 = 6b_0 + 42b_1 + 302b_2 \ldots (1)$$

$$236 = 42b_0 + b_1 434 + 2824b_2 \ldots\ldots\ldots (2)$$

$$1846 = 302b_0 + 2824b_1 + 19060b_2 \ldots (3)$$

Ans.

Example 6.22. Fit an equation of the form $y = ab^x$ to the following data

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

(November 2006 Set 2)
(Supple. Feb. 2010 Set.1)

Solution Given curve is $y = ab^x$

Taking logarithms of both sides

$$\log_{10} y = \log_{10} a + x \log_{10} b$$

$$Y = A + BX$$

Where Y = log y, X = x, A = log a, B = log b

x=X	y	Y=$\log_{10}$y	X^2	XY
2	144	2.1584	4	4.3168
3	172.8	2.2375	9	6.7125
4	207.4	2.3168	16	9.2672
5	248.8	2.3959	25	11.9795
6	298.5	2.4749	36	14.8494
ΣX = 20	ΣY = 1071.5	ΣY = 11.5835	$\Sigma X^2 = 90$	ΣXY = 47.1254

Normal equations are

$$\sum Y = nA + B\sum X \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \ldots(2)$$

11.5835 = 5A + 20B .... (1)

47.1254 = 20A + 90B ...... (2)

Eq. (1) multiplied by 4

46.334 = 20A + 80B

47.1254 = 20A + 90B

-0.7914 = - 10B

B = .07914

B = log b = .07914

b = 1.199

5A + 20B = 11.5835

5A + 20 × (0.07914) = 11.5835

5A = 11.5835 – 1.5828

5A = 10.0007

A = 2.00014

A = log a = 2.00014

Taking antilog

a = 100

$y = 100\,(1.199)^x$ **Ans.**

Example 6.23. Obtain a relation of the term $y = a(b)^x$ for the following data by the method of least squares.

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	17.4

(Supple. Feb 2007 Set 4)

Solution

Given curve is $y = a\,(b)^x$

Taking logarithms of both sides

$$\log_{10} y = \log_{10} a + x\log_{10} b$$

Y = A + BX

Where $Y = \log_{10} y$, $X = x$, $A = \log_{10} a$, $B = \log_{10} b$

x = X	y	Y=$\log_{10}$y	X^2	XY
2	8.3	.9190	4	1.8382
3	15.4	1.1872	9	3.5616
4	33.1	1.5198	16	6.0792
5	65.2	1.8142	25	9.0710
6	17.4	2.1052	36	12.6312
ΣX= 20	Σy= 139.4	ΣY= 7.5455	ΣX^2 =90	ΣXY= 33.1812

Normal equations are

$$\sum Y = nA + B\sum X \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \ldots(2)$$

7.5455 = 5A + 20B .... (1)

33.1812 = 20A + 90B .... (2)

Eq. (1) multiplied by 4

30.182 = 20A + 80B

33.1812 = 20A + 90B

-2.9992 = - 10B

B = 0.29992

From eq. (1)

7.5455 = 5A + 20B

7.5455 = 5A + 20 (0.29992)

7.5455 = 5A + 5.9984

5A = 1.5471

A = 0.30942

A = log a = 0.30942

A = 2.03, B = log b = 0.29992

$y = 2.03\ (1.994)^x$ **Ans.**

ıns.

ving data by the

Feb 2007 Set 4)

Example 6.24. The following data relate to the marks of 10 students in the internal test and the university examination for the maximum of 50 each

(Supple. Feb 2007 Set No.3)
(Supple. Feb. 2010 Set.1)

Internal marks (x)	25	28	30	32	35	36	38	39	42	45
University marks (y)	20	26	29	30	25	18	26	35	35	46

Find the coefficient of correlation and the two lines of regression

Solution

x	y	x^2	y^2	xy
25	20	625	400	500
28	26	784	676	728
30	29	900	841	870
32	30	1024	900	960
35	25	1225	625	875
36	18	1296	324	648
38	26	1444	676	988
39	35	1521	1225	1365
42	35	1764	1225	1470
45	46	2025	2116	2070
$\Sigma x=350$	$\Sigma y=290$	$\Sigma x^2=12608$	$\Sigma y^2=9008$	$\Sigma xy=10474$

$$r=\frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$S_{xx}=\sum_{i=1}^{n}x_i^2-\left(\sum_{i-1}^{n}x_i\right)^2\Big/n=12608-\frac{(350)^2}{10}=12608-12250=358$$

$$S_{yy}=\sum_{i=1}^{n}y_i^2-\left(\sum_{i-1}^{n}y_i\right)^2\Big/n=9008-\frac{(290)^2}{10}=9008-8410=598$$

$$S_{xy}=\sum_{i=1}^{n}x_iy_i-\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right)\Big/n=10974-\frac{350\times 290}{10}=10974-10150=324$$

$$r=\frac{324}{\sqrt{358\times 598}}=\frac{324}{\sqrt{214084}}=\frac{324}{462.69}=0.700$$

The line of reg

The line of regre
$y-\bar{y}=r\frac{\sigma_y}{\sigma_x}($
$\sigma_x^2=\frac{1}{n}\sum x^2$
$=\frac{1}{10}\,12608-$
$=1260.8-12$
$\sigma_x^2=35.8$
$\sigma_x=5.98$

$y-29=0.7$

After solving
y - .905x = -2.6

$x-35=0.7$

After solving
x – 0.542y = 1

Example 6.25. Calcu their sons

Heights of
Heights of

Solution

Heights of father (x)	I
65	
66	
67	
67	
68	
69	
70	
72	
$\Sigma x = 482$	

The line of regression of y and x

The line of regression of y on x	The line of regression of x on y
$y-\bar{y}=r\dfrac{\sigma_y}{\sigma_x}(x-\bar{x})$	$x-\bar{x}=r\dfrac{\sigma_x}{\sigma_y}(y-\bar{y})$
$\sigma_x^2=\dfrac{1}{n}\sum x^2-\bar{x}^2$	$\sigma_y^2=\dfrac{1}{n}\sum y^2-\bar{y}^2$
$=\dfrac{1}{10}\ 12608-(35)^2$	$=\dfrac{1}{10}9008-(29)^2$
$=1260.8-1225$	$\sigma_y^2=9008-841$
$\sigma_x^2=35.8$	$\sigma_y^2=59.8$
$\sigma_x=5.98$	$\sigma_y=7.73$

$$y-29=0.7\times\frac{7.73}{5.98}(x-35)$$

After solving

y - .905x = -2.675 **Ans.**

$$x-35=0.7\times\frac{5.98}{7.73}(y-35)$$

After solving

x – 0.542y = 19.282 **Ans.**

Example 6.25. Calculate the correlation coefficient between the heights of fathers and their sons

Heights of father (x)	65	66	67	67	68	69	70	72
Heights of sons (y)	67	68	65	68	72	72	69	71

(Supple. Nov. /Dec. 2004 Set 4)

Solution

Heights of father (x)	Heights of sons (y)	x^2	y^2	xy
65	67	4225	4489	4355
66	68	4356	4624	4488
67	65	4489	4225	4355
67	68	4489	4624	4556
68	72	4624	5184	4896
69	72	4761	5184	4968
70	69	4900	4761	4830
72	71	5184	5041	5112
$\Sigma x = 482$	$\Sigma y = 552$	$\Sigma x^2 = 37028$	$\Sigma y^2 = 38132$	$\Sigma xy = 37560$

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 37028 - \frac{(482)^2}{8} = 37028 - \frac{232324}{8} = 7987.5$$

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/ n = 38132 - (552)^2/_8 = 38132 - 304704/_8 = 44$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = 37560 - (482)(552)/_8 = 37560 - 266064/_8 = 4302$$

$$r = \frac{4302}{\sqrt{(7987.5)44}} = \frac{4302}{\sqrt{351450}}$$

$$r = \frac{4302}{592.83} = 7.256 \textit{ Ans.}$$

60b + 600c = 75 …. (4)

5400b + 56772 c = 6513 …. (5)

Eq. (4) multiplied by 90

5400b + 54000 c = 6750

5400b + 56772c = 6513

\- 2772 c = 237

$$c = \frac{237}{2772}$$

$$c = -0.08549$$

60b + 600c = 75

60b + 600 X (- 0.08549) = 75

60b – 51.2987 = 75

60b = 126.2987

b = 2.1049

9a + 45b + 285c = 80

9a + 45(2.1049) + 285 (-0.08549) = 80

9a + 94.7205 – 24.36465 = 80

9a = 9.64415

a =

y =

y =

y =

Example 6.26. F

value of y when x

Solution

x
0
5
10
15
20
Σx =

Assume that

Its normal e

$\sum y$

$\sum x$

80 = $5a_0$

1035 = $50a_0$

Eq. 1) multi

800 = $50a_0$ +

1035 = $50a_0$

\- 235 =

$$a_1 = \frac{235}{250}$$

$$a_1 = 0.94$$

$$a = 1.07157$$

$$y = a + bx + cx^2$$

$$y = 1.07157 + 2.1049x + (-0.08549)x^2$$

$$y = 1.07157 + 2.1049x - 0.08549x^2$$

Ans.

Example 6.26. Fit a straight line $y = a_0 + a_1 x$ for the following data and estimate the value of y when x = 25

x	0	5	10	15	20
y	7	11	16	20	26

(Supple. Feb. 2007 Set 1)
(Supple. Feb. 2010 Set 1, 2, 3, 4)

Solution

x	y	xy	x2	y^2
0	7	0	0	49
5	11	55	25	121
10	16	160	100	256
15	20	300	225	400
20	26	520	400	676
$\Sigma x = 50$	$\Sigma y = 80$	$\Sigma xy = 1035$	$\Sigma x^2 = 750$	$\Sigma y^2 = 1502$

Assume that the least squares straight line of y on x is $y = a_0 + a_1 x$
Its normal equations are

$$\Sigma y = Na_0 + a_1 \Sigma x$$

$$\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2$$

$80 \quad = 5a_0 + a_1\, 50$............1)
$1035 = 50a_0 + 750a_1$............2)

Eq. 1) multiplied by 10
$800 = 50a_0 + 500a_1$
$1035 = 50a_0 + 750a_1$

$-235 = \quad -250a_1$

$$a_1 = \frac{235}{250}$$

$$a_1 = 0.94$$

$80 = 5\ a_0 + a_1\ 50$

$80 = 5\ a_0 + 0.94 \times 50$

$80 = 5\ a_0 + 47$

$5\ a_0 = 33$

$a_0 = 6.6$

$y = 6.6 + 0.94\ x$ **Ans.**

Value of y when $x = 25$

$y = 6.6 + 0.94 \times 25$

$= 6.6 + 23.5$

$y = 30.1$ **Ans.**

Example 6.27. The following data pertain to the number of computer jobs per day and the central processing unit (CPU) time required

Number of jobs (x)	1	2	3	4	5
CPU time (y)	2	5	4	9	10

(Supple.Nov./Dec.2004 Set 3)
(Supple. April/May 2005 Set 1)
(Supple.Nov./Dec.2005 Set 3)
(Supple. Feb. 2007 Set 2)

a. Obtain a least squares fit of a line to the observations on CPU time.
   Use the equation of the least squares line to estimate the mean CPU time at $x = 3.5$

b. Construct a 95% confidence interval for α.

c. Test the null hypothesis $B = 2$ against the alternative hypothesis $B>2$ at the 0.05 level of significance

Solution

No. of jobs x	CPU time y	xy	x^2	y^2
1	2	2	1	4
2	5	10	4	25
3	4	12	9	16
4	9	36	16	81
5	10	50	25	100
$\Sigma x=15$	$\Sigma y=30$	$\Sigma xy=110$	$\Sigma x^2=55$	$\Sigma y^2=226$

For these n = 5 pairs (x_i, y_i), we first calculate

$$\sum_{i=1}^{n} x_i = 15$$

$$\sum_{i=1}^{n} y_i = 30$$

$$\sum_{i=1}^{n} y_i^2 = 226$$

$$\sum_{i=1}^{n} x_i^2 = 55$$

$$\sum_{i=1}^{n} x_i y_i = 110$$

And we obtain

$$S_{xx} = \sum_{i=1}^{n} x_i^{\,2} - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 55 - (15)^2/5 = 55 - \frac{225}{5} = 10$$

$$S_{yy} = \sum_{i=1}^{n} y_i^{\,2} - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/ n = 226 - (30)^2/5 = 226 - \frac{900}{5} = 46$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = 110 - \frac{15 \times 30}{5} = 110 - 90 = 20$$

Consequently

$$b = \frac{S_{xy}}{S_{xx}} = \frac{20}{10} = 2$$

And then

$$\begin{aligned} a &= \overline{y} - b\overline{x} \\ &= 6 - 2 \times 3 \\ &= 6 - 6 = 0 \end{aligned}$$

Thus, the equation of the straight line that best fits the given data in the sense of least squares is

$$\hat{y} = a + bx$$

$$\hat{y} = 0 + 2x$$

$$\hat{y} = 2x$$ **Ans.**

b. and for x = 3.5, mean CPU time is

$$\hat{y} = 2 \times 3.5 = 7$$

Ans.

c. Construct a 95% confidence interval forα, 95% confidence limits for α are

$$a \pm t_{\alpha/2} S_{\varepsilon} \sqrt{\frac{S_{xx} + (n\bar{x})^2}{n S_{xx}}}$$

Or

$$a \pm t_{\alpha/2} S_{\varepsilon} \sqrt{\frac{1}{n} + \frac{(\bar{x})^2}{S_{xx}}}$$

$$S_{\varepsilon}^2 = \frac{S_{yy} - (Sxy)^2 / S_{xx}}{n-2}$$

$$= \frac{46 - (20)^2 / 10}{5-2}$$

$$= \frac{46 - 400/10}{3}$$

$$S_{\varepsilon} = 1.414$$

$$1 - \alpha = 0.95$$

$$1 - .95 = \alpha$$

$$0.05 = \alpha$$

$$t_{\alpha/2} = t_{0.05/2}$$

$$t_{\alpha/2} = t_{0.025} = 3.12$$

n = 5

$$n - 2 = 5 - 2 = 3$$

Now confidence limits for α

$$a \pm t_{\alpha/2} S_{\varepsilon} \sqrt{\frac{1}{n} + \frac{(\bar{x})^2}{S_{xx}}}$$

$$= \pm\, 4.4993\sqrt{0.2 + 0.9}$$

$$= \pm 4.4993\sqrt{1.1}$$

$$= \pm 4.7189$$

Ans.

d. Test the null hypothesis B = 2 against the alternative hypothesis B>2 at the 0.05 level of significance

1. Null hypothesis: B = 2

2. Alternative
3. Level of si
4. Critical reg

Accept N.I
with 3 D.O

Thus accep

Where $t =$

5. Computati

$S_{\varepsilon} = 1.414,$

$t = \frac{(2-2}{1.414}$

6. Decision:

Accept N.F

0 < 2.353 v

Example 6.28. A che
the efficiency of an ex

Extraction time (mir
Extraction efficiency

Fit a straight line to th
the extraction efficienc

Solution

Extraction time (min.) x	ef
27	
45	
41	
19	
35	
39	
19	
49	
15	
31	
Σx=320	

2. Alternative hypothesis: B > 2
3. Level of significance: $\alpha = 0.05$
4. Critical region: Right one tailed test.

   Accept N.H if $t < t_\alpha = t_{0.05}$ with n-2 degrees of freedom. From t -table $t_{0.05}$ with 3 D.O.F is 2.353.

   Thus accept NH if $t < t_\alpha = 2.353$

   Where $t = \dfrac{(b-B)}{S_\varepsilon}\sqrt{S_{xx}}$

5. Computation: n = 5, b = 2, B = 2

   $S_\varepsilon = 1.414$, $S_{xx} = 10$

   $$t = \frac{(2-2)}{1.414}\sqrt{10} = 0$$

6. Decision:

   Accept N.H $t < t_\alpha$

   $0 < 2.353$ with 3 d.o.f. **Ans.**

Example 6.28. A chemical company, wishing to study the effect of extraction time on the efficiency of an extraction operation, obtained the data shown in the following table:

Extraction time (min.) x	27	45	41	19	35	39	19	49	15	31
Extraction efficiency(%) y	57	64	80	46	62	72	52	77	57	68

Fit a straight line to the given data by the method of least squares and use it to predict the extraction efficiency one can expect when the extraction time is 35 minutes.

(Nov.2006 Set 1)
(NR Supple.Feb.2010 Set.1, 2, 3, 4)
(R05Supple.Feb.2010 Set.3)

Solution

Extraction time (min.) x	Extraction efficiency (%) y	xy	x^2	y^2
27	57	1539	729	3249
45	64	2880	2025	4096
41	80	3280	1681	6400
19	46	874	361	2116
35	62	2170	1225	3844
39	72	2808	1521	5184
19	52	988	361	2704
49	77	3773	2401	5929
15	57	855	225	3249
31	68	2108	961	4624
$\Sigma x=320$	$\Sigma y=635$	$\Sigma xy=21275$	$\Sigma x^2=11490$	$\Sigma y^2=41395$

$y = a + bx$

For these n = 10 pairs (x_i, y_i), we first calculate

$$\sum_{i=1}^{n} x_i = 320$$

$$\sum_{i=1}^{n} y_i = 635$$

$$\sum_{i=1}^{n} y_{,i}^2 = 41395$$

$$\sum_{i=1}^{n} x_1^2 = 11490$$

$$\sum_{i=1}^{n} x_i y_i = 21275$$

And then we obtain

$$S_{xx} = \sum_{i=1}^{n} x_i^{\,2} - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 11490 - \frac{(320)^2}{10} = 11490 - \frac{10240}{10} = 1250$$

$$S_{yy} = \sum_{i=1}^{n} y_i^{\,2} - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/ n = 41395 - \frac{(635)^2}{10} = 41395 - 40322.5 = 1072.5$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i^{\,2} - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = 21272 - \frac{(320)(635)}{10} = 21275 - 20320 = 95$$

Consequently

$$b = \frac{S_{xy}}{S_{xx}} = \frac{955}{1250} = 0.764$$

And then

$a = \overline{y} = b\overline{x} = 63.5 - 0.764(32) = 63.5 - 24.448 = 39.052.$

Thus, the equation of the straight line that best fits the given data in sense of least squares is

$\hat{y} = a + bx$

$\hat{y} = 39.052 + (0.764)x$

And for x = 35 we predict that the extraction efficiency will be

$\hat{y} = 39.052 + 0.764(35) = 39.052 + 26.74 = 65.792$ **Ans.**

Example 6.29.

X
Y

Solution

$y = ae^{b}$

$\log_{10} y$

$\log_{10} y$

$\log_{10} y$

$\log_{10} y$

$Y = A +$

$Y = \log$

x	y
0	20
1	30
2	52
3	77
4	135
5	211
6	326
7	550
8	1052

Normal e

$\Sigma Y = n$

$\Sigma XY =$

4.6051 = 9A +

11.1774 = 15.6

71.9832

100.593

-28.6098

Example 6.29. Fit the curve $y = ae^{bx}$ to the following data.

X	0	1	2	3	4	5	6	7	8
Y	20	30	52	77	135	211	326	550	1052

(Supple.Nov./Dec.2004)

Solution

$$y = ae^{bx}$$
$$\log_{10} y = \log_{10}\left(ae^{bx}\right)$$
$$\log_{10} y = \log_{10} a + \log_{10} e^{bx}$$
$$\log_{10} y = \log_{10} a + bx \log_{10} e$$
$$\log_{10} y = \log_{10} a + bx(0.4342)$$
$$Y = A + BX$$
$$Y = \log_{10} y, A = \log_{10} a,\ B = b, X = x(0.4342)$$

x	y	X=x(0.4342)	Y = $\log_{10}$y	X^2	XY
0	20	0	0	0	0
1	30	0.4342	0	0.1885	0
2	52	0.8684	0.3010	0.7541	0.2613
3	77	1.3026	0.4771	1.6967	0.6214
4	135	1.7368	0.6020	3.0164	1.0455
5	211	2.171	0.6989	4.7132	1.5173
6	326	2.6052	0.7781	6.7870	2.0271
7	550	3.0394	0.8450	9.2379	2.5682
8	1052	3.4736	0.9030	12.0658	3.1366
		ΣX =15.6312	ΣY= 4.6051	ΣX^2 =38.4596	ΣXY = 11.1774

Normal equations are

$$\Sigma Y = nA + B\Sigma X$$
$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

$4.6051 = 9A + 15.6312B$ .... (1)

$11.1774 = 15.6312A + 38.4596B$.................. (2)

$71.9832 = 140.6808A + 244.3344B$

$100.593 = 140.6808A + -346.1364B$

$-28.6098 = \quad -101.802B$

) – = 1250

2.5 = 1072.5

1275 – 20320 = 955

a in sense of least

Ans.

$$B = \frac{28.6098}{101.802} = 0.28103$$

4.6051 = 9A + (15.6312) (.2810)

4.6051 = 9A + 4.3928

2122 = 9A

A = 0.023

$A = \log_{10} a \qquad B = b$

$0.023 = \log_{10} a \qquad 0.28103 = b$ **Ans.**

$10^{0.023} = a$

1.0543 = a. **Ans.**

Example 6.30. The following are the measurements of the air velocity and evaporation coefficient of burning fuel droplets in air impulse engine.

Air velocity (x)	20	60	100	140	180	220	260	300	340	380
Evaporation coefficient (y)	.18	.37	.35	.78	.56	.75	1.18	1.36	1.17	1.65

(Supple Nov./Dec.2005 Set 2)
(R05Supple.Feb.2010 Set.2)

Fit a straight line to these data by the method of least squares, and use it to estimate the evaporation coefficient of a droplet when the air velocity is 190 cm/S.

Solution

Air velocity x	Evaporation Coefficient y	xy	x^2	y^2
20	0.18	3.6	400	0.0324
60	0.37	22.2	3600	0.1369
100	0.35	35.0	10000	0.1225
140	0.78	109.2	19600	0.6084
180	0.56	100.8	32400	0.3136
220	0.75	165.0	48400	0.5625
260	1.18	306.8	97600	1.3924
300	1.36	408.0	90000	1.8496
340	1.17	397.8	115600	1.3689
380	1.65	627.0	144400	2.7225
Σx=2000	Σy=8.35	Σxy=2175.4	Σx^2 =532000	Σy^2= 9.109

For these

$\sum_{i=1}^{n} x_i =$

$\sum_{i=1}^{n} y_i =$

$\sum_{i=1}^{n} y_i^2 =$

$\sum_{i=1}^{n} x_1^2 =$

$\sum_{i=1}^{n} x_i y_i$

And then we ca

$S_{xx} = \sum_{i=1}^{n} x_i^2 -$

$S_{yy} = \sum_{i=1}^{n} y_i^2 -$

$S_{xy} = \sum_{i=1}^{n} x_i y_i$

$b = \frac{S_{xy}}{S_{xx}}$

$a = \bar{y} -$

= 0.069

The equa

$\hat{y} = 0.0$

For x = 1

Estimate the ev

$\hat{y} = 0.0$

$\hat{y} = 0.8$

ıd evaporation

340	380
1.17	1.65

ec.2005 Set 2)
eb.2010 Set.2)
to estimate the

y^2
0.0324
0.1369
0.1225
0.6084
0.3136
0.5625
1.3924
1.8496
1.3689
2.7225
Σy^2= 9.109

For these n = 10 pairs

$$\sum_{i=1}^{n} x_i = 2000$$

$$\sum_{i=1}^{n} y_i = 8.35$$

$$\sum_{i=1}^{n} y_{,i}^2 = 9.1047$$

$$\sum_{i=1}^{n} x_1^2 = 532000$$

$$\sum_{i=1}^{n} x_i y_i = 2175.4$$

And then we can obtain

$$S_{xx} = \sum_{i=1}^{n} x_i^{\,2} - \left(\sum_{i-1}^{n} x_i\right)^2 \Big/ n = 332000 - (2000)^2 /_{10} = 132000$$

$$S_{yy} = \sum_{i=1}^{n} y_i^{\,2} - \left(\sum_{i-1}^{n} y_i\right)^2 \Big/ n = 9.1097 - (8.35)^2 /_{10} = 2.13745$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = 2175.40 - (2000)(8.35)/_{10} = 505.40$$

$$b = \frac{S_{xy}}{S_{xx}} = \frac{505.40}{132000} = 0.00383$$

$$a = \bar{y} - b\bar{x} = \frac{8.35}{10} - 0.00383\left(\frac{2000}{10}\right)$$

$= 0.069$

The equation of the straight line that best fit the given data

$\hat{y} = 0.069 + 0.00383x$ **Ans.**

For x = 190,

Estimate the evaporation coefficient of a droplet when the air velocity is x = 190 cm/S.

$\hat{y} = 0.069 + 0.00383 \times (190)$

$\hat{y} = 0.80$ **Ans.**

Example 6.31. Fit the model $y = ax^b$ to the following data

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.10	6.80	7.50

Solution

Given curve is $y = ax^b$

Taking logarithms of both sides

$\log_{10} y = \log_{10} a + b \log_{10} x$

$Y = A + BX$

Where Y = log y

$A = \log a \quad and \quad X = \log x$

Then the normal equations are

$\sum Y = nA + B\sum X$........(1)

$\sum XY = A\sum X + B\sum X^2$.......(2)

x	y	X=$\log_{10}$x	Y=$\log_{10}$y	X^2	XY
1	2.98	0	.4742	0	0
2	4.26	.3010	.6294	.0906	.1894
3	5.21	.4771	.7168	.2276	.3420
4	6.10	.6021	7853	.3625	.4728
5	6.80	.6990	.8325	.4886	.5819
6	7.50	.7782	.8751	.6056	.6810
		ΣX=2.8574	ΣY=4.3133	ΣX^2=1.7749	ΣXY=2.2671

By the normal equations

4.3133 = 6A + B 2.8574 ....(1)

2.2671 = 2.8574 A + 1.7749 B ....(2)

Eq. (1) multiplied by 2.8574 $\quad$ 12.3248 = 17.1444 A + 8.1697 B

Eq. (2) multiplied by 6 $\quad$ 13.6026 = 17.1444A + 10.6494 B

- 1.2778 = $\quad$ - 2.4847 B

By the equat

4.3133 = 6A

4.3133 = 6A

4.3133 = 6A

6A = 2.8438

A = 0.47397

A = log a

0.47397 = l

$10^{.97397} =$

$a = 2.9783$

$B = b = 0.5$

$y = ax^b$

$y = 2.9783$

Example 6.32. Fit

Solution

Given curve is $y =$

Normal equations ar

$\sum_{i=1}^{n} y_1 = na_0$

$\sum_{i=1}^{n} x_i y_i = a$

$\sum_{i=1}^{n} x_i^2 y_i =$

$$B = \frac{1.2778}{2.4847} = 0.5142$$

By the equation (1)

4.3133 = 6A + 2.8574 B

4.3133 = 6A + (2.8574) (0.5142)

4.3133 = 6A + 1.4694

6A = 2.84383

A = 0.47397

A = log a

$0.47397 = \log_{10} a$

$10^{.97397} = a$

$a = 2.9783$

$B = b = 0.5142$

$y = ax^b$

$y = 2.9783\, x^{0.5142}$ **Ans.**

Example 6.32. Fit the parabola $y = a_0 + a_1 x + a_2 x^2$ for the following data

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(Supple Feb.2007 Set 3)
(April./May.2005 Set 4)
(R05Supple.Feb.2010 Set.3)

Solution

Given curve is $y = a_0 + a_1 x + a_2 x^2$

Normal equations are

$$\sum_{i=1}^{n} y_1 = na_0 + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_1^{\,2}$$

$$\sum_{i=1}^{n} x_i y_i = a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^{\,2} + a_2 \sum_{i=1}^{n} x_i^{\,3}$$

$$\sum_{i=1}^{n} x_i^{\,2} y_i = a_0 \sum_{i=1}^{n} x_i^{\,2} + a_1 \sum_{i=1}^{n} x_i^{\,3} + a_2 \sum_{i=1}^{n} x_i^{\,4}$$

XY
0
.1894
.3420
.4728
.5819
.6810
ΣXY=2.2671

x	y	xy	x^2	x^3	x^4	x^2y
0	1	0	0	0	0	0
1	1.8	1.8	1	1	1	1.8
2	1.3	2.6	4	8	16	5.2
3	2.5	7.5	9	27	81	22.5
4	6.3	25.2	16	64	256	100.8
$\Sigma x = 10$	$\Sigma y = 12.9$	$\Sigma xy = 37.1$	$\Sigma x^3 = 100$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma x^2y = 130.3$

From the normal equations

$$12.9 = 5a_0 + 10a_1 + 30a_2 \;\ldots\ldots\ldots (1)$$

$$37.1 = 10a_0 + 30a_1 + 100a_2 \;\ldots\ldots(2)$$

$$130.3 = 30a_0 + 100a_1 + 354a_2 \ldots (3)$$

Eq. (1) multiplied by 2

$$25.8 = 10a_0 + 20a_1 + 60a_2$$

$$37.1 = 10a_0 + 30a_1 + 100a_2$$

$$-11.3 \qquad -10a_1 - 40a_2 \;\ldots\ldots (4)$$

Eq. (2) multiplied by 3

$$111.3 = 30\,a_0 + 90a_1 + 300a_2$$

$$130.3 = 30\,a_0 + 100a_1 + 354a_2$$

$$-19 \;=\; -10a_1 \;\; -54a_2 \;\ldots\ldots (5)$$

From (4) and (5)

$$a_2 = 0.55$$
$$a_1 = -1.07$$

From the equation ......(1)

$$a_0 = 1.42$$

$$y = 1.42 - 1.07x + 0.55x^2$$ **Ans.**

Example 6.33. Use the formula $x = \dfrac{\sigma_x^2 + \sigma_y^2 + \sigma_{xy}^2}{2\sigma_x\,\sigma_y}$ to compute the correlation coefficient to the following data

x	62	56	36	66	25	75	82	78
y	58	44	51	58	60	68	62	84

Solution

x
62
56
36
66
25
75
82
78
$\Sigma x = 480$

Now

$\sigma_x^2 = \dfrac{\Sigma x^2}{N} - ($

$\sigma_y^2 = \dfrac{\Sigma y^2}{N} - ($

$\sigma_{xy}^2 = \Sigma($

$=$

By formula

$r = \dfrac{\sigma_x^2 +}{2}$

$= \dfrac{27333(}{421.}$

Curve fitting

Least square str

1) Fit a straig

X (

Y :

[Hint: N=5,

Y=28/1

	x^2y
	0
	1.8
	5.2
	22.5
	100.8
4	$\Sigma x^2y = 130.3$

Ans.

e the correlation

Solution

x	y	x^2	y^2	xy	$(xy)^2$
62	58	3844	3364	3596	12931216
56	44	3136	1936	2464	6071296
36	51	1296	2601	1836	3370896
66	58	4356	3364	3828	14653584
25	60	625	3600	1500	2250000
75	68	5625	4624	5100	26010000
82	62	6724	3844	5084	25847056
78	84	6084	7056	6552	42928704
$\Sigma x = 480$	$\Sigma y = 485$	$\Sigma x^2 = 31690$	$\Sigma y^2 = 30389$	$\Sigma xy = 29960$	$\Sigma(xy) = 134062752$

Now

$$\sigma_x^2 = \frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2 = \frac{31690}{8} - \left(\frac{480}{8}\right)^2 = 3961.25 - 3600 = 361.25$$

$$\sigma_y^2 = \frac{\sum y^2}{N} - \left(\frac{\sum y}{N}\right)^2 = \frac{30389}{8} - \left(\frac{485}{8}\right)^2 = 3798.62 - 3675.390 = 123.229$$

$$\sigma_{xy}^2 = \frac{\sum (xy)^2}{N} - \left(\frac{\sum xy}{N}\right)^2 = \frac{134062752}{8} - \left(\frac{29960}{8}\right)^2$$

$$= 16757844 - 14025025 = 2732819$$

By formula

$$r = \frac{\sigma_x^2 + \sigma_y^2 + \sigma_{xy}^2}{2\sigma_x \sigma_y} = \frac{361.25 + 123.229 + 2732819}{2 \times 19 \times 11.10}$$

$$= \frac{2733303.47}{421.83} = 6479.63$$

Ans.

EXERCISE

Curve fitting

Least square straight line

1) Fit a straight line y=a+bx to the following data by the method of least squares

X	0	2	5	6	8
Y	1	3	2	5	3

[Hint: N=5, $\sum x=21$, $\sum y=14$, $\sum x^2=129$, $\sum xy=70$

Y=28/17+ (14/51) x **Ans.]**

2) Fit a least square straight line to the following data.

X	-3	-2	-1	0	1	2	3
Y	1	1	2	2	3	3	4

[**Hint**: N=7, $\sum x=0$, $\sum Y=16$ $\sum x^2=28$, $\sum xy=14$

Y=2 + (0.5) x **Ans.**]

Least squares quadratic curve:

3) Fit a least square parabola to the following data.

X	0.0	0.1	0.2	0.3	0.4
Y	1.26	1.50	0.56	0.24	0.39

[**Hint:** n=5, $\sum x=1.0$, $\sum x^2=0.3$, $\sum x^3=0.1$, $\sum x^4=0.0354$, $\sum Y=3.95$, $\sum xy=0.49$, $\sum x^2y=0.1214$

$Y=1.41+(3.42)x+1.07x^2$ **Ans.**]

4) Fit a least squares quadratic curve to the following data

X	1	2	3	4	
Y	1.9	1.7	1.8	1.4	Estimate y (1.75)

[**Hint:** n=4, $\sum x = 10$, $\sum x^2 = 30$, $\sum x^3 = 100$, $\sum x^4 = 354$, $\sum Y = 6.8$, $\sum xy = 16.3$, $\sum x^2y = 47.3$

$Y=1.59+(0.32)x+(-0.092)x^2$, y = 1.868 at x=1.75 **Ans.**]

Curvilinear Regression
Exponential curve

5) Fit an exponential curve of the form $y=ae^{bx}$ for the following data

x	1	2	3	4
y	11	12	13	14

6) Fit an exponential curve to the following data

X	1	2	3	5	
Y	3	4	5	3	Estimate y (4.5)

[**Hint:** n=4, $\sum x=\log x=3.4011$, $\sum Y=\log y=5.1281$, $\sum x^2=4.27746$, $\sum xy=4.4888$

$Y=(3.50119)x^{1.05447}$, y=17.10049(at x=4.5) **Ans.**]

7) Estimate y at x=4, by fitting a least square curve of the form y=b/x(x-a) to the following data:

X	2.1	3.1	4.1	5.1	6.1
Y	0.35	0.40	0.35	0.40	0.35

[**Hint:** $\sum x=20.5$, $\sum Y=13.5113$, $\sum x^2=94.05$, $\sum x^3=467.605$, $\sum xy=26.8$,

a=5.262180, b=-1.0186, y=0.201754(at x=4) **Ans.**]

1. The residual
regression lin

a) $SSE = \sum_{i=1}^{n}$

c) $SSE = \sum_{i=1}^{n}$

Ans. (a)

2. The least squ
α and β are

a) $a = \frac{\sum_{i=1}^{n} x_i}{n}$,

b) $a = \frac{\sum_{i=1}^{n} y_i}{}$

c) $a = \frac{\sum_{i=1}^{n} (x-}{\sum_{i=1}^{n} (}$

Ans. (b)

3. The regressio

$\sum_{i=1}^{30} x_i = 1104$

a) $y = 3.225$

c) $y = 4.212$

Ans. (c)

4. The regressio

$n=10, \sum x$

a) $y = 0.069$

c) $y = 0.069$

Ans. (d)

OBJECTIVE TYPE QUESTIONS

1. The residual sum of squares or sum of squares of the errors about the regression line (SSE) given by

a) $SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$ b) $SSE^2 = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$

c) $SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (a + bx_i - y_i)^2$ d) $SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (a + bx_i - y_i)$

Ans. (a)

2. The least squares estimates a and b of Regression the regression coefficients α and β are computed from

a) $a = \dfrac{\sum_{i=1}^{n} x_i}{n}, \; b = \dfrac{\sum_{i=1}^{n} y_i}{n}$

b) $a = \dfrac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n}, \; b = \dfrac{\sum_{i=1}^{n} (x - \bar{x})(y - \bar{y})}{\sum_{i=1}^{n} (x - \bar{x})^2}$

c) $a = \dfrac{\sum_{i=1}^{n} (x - \bar{x})(y - \bar{y})}{\sum_{i=1}^{n} (x - \bar{x})^2}, \; b = \dfrac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n}$ d) None of the above

Ans. (b)

3. The regression line for the given data

$\sum_{i=1}^{30} x_i = 1104$, $\sum_{i=1}^{30} y_i = 1124$, $\sum_{i=1}^{30} x_i y_i = 41{,}355$, $\sum_{i=1}^{30} x_i^2 = 41{,}086$,

a) $y = 3.2255 + 0.01787x$ b) $y = 4.0015 + 1.2534x$

c) $y = 4.2126 + 0.01787x$ d) $y = 2.2126 + 0.01787x$

Ans. (c)

4. The regression line for the given data

$n = 10$, $\sum x = 2000$, $\sum y = 8.35$, $\sum xy = 2175.40$, $\sum x^2 = 532{,}000$,

a) $y = 0.069 + 0.0058x$ b) $y = 0.069 + 0.158x$

c) $y = 0.069 + 0.235x$ d) $y = 0.069 + 0.0038x$

Ans. (d)

5. Two or more independent variables in a regression equation, it is known as
   a) Curve Linear Regression b) Multiple Regression
   c) Linear Regression d) None of the above

Ans. (b)

6. The paired values plotted on a graph marked by dot is called
   a) Bar Diagram b) Statistical Diagram
   c) Scatter Diagram d) None of the above

Ans. (c)

7. The formula for Correlation Coefficient r is given by
   a) $r = \dfrac{S_{xy} S_{xx}}{S_{yy}}$ b) $r = \dfrac{S_{xy} S_{yy}}{S_{xx}}$
   c) $r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ d) None of the above

Ans. (c)

8. The equation of second degree parabola is
   a) $y = a + bx + c$ b) $y = a + bx + cx^2$
   c) $y = a + bx$ d) $y = ax + b$

Ans. (b)

9. Sample Correlation Co-efficient for the given data
   n = 12, $S_{xy} = 93$, $S_{xx} = 126$, $S_{yy} = 123$
   a) 0.08562 b) 0.7070 c) 0.7120 d) 0.7470

Ans. (d)

10. The least square line that fits a set of sample points is given by y –
    Where the constants a and b are determined by the normal equations. normal equations are
    a) $\sum y = na + \sum x$..........(1) $\sum xy = a\sum x + b\sum x^2$..........(2)
    b) $\sum y = a + b\sum x$..........(1) $\sum xy = a\sum x + b\sum x^2$..........(2)
    c) $\sum y = na + b\sum x$..........(1) $\sum xy = a\sum x + b\sum x^2$..........(2)
    d) $\sum y = n + b\sum x$..........(1) $\sum xy = a\sum x + b\sum x^2$..........(2)

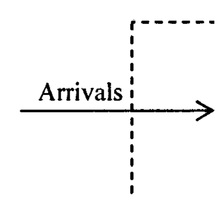
Ans. (d)

"The basic idea behi

In our daily li
window. If the arriv
this situation they h
the customers and th

Examples of w

Jobs waiting f
board, patients arriv
getting ticket near th
repairing, arriving v
vehicles, machines e



UNIT-7

QUEUING THEORY

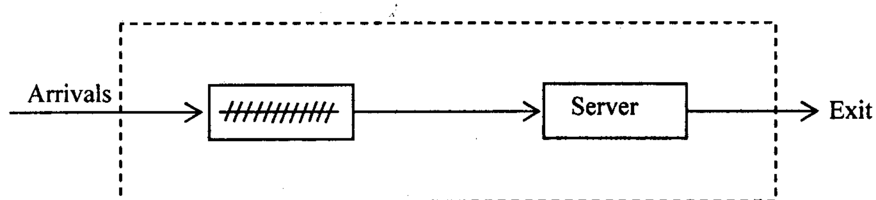
"The basic idea behind teaching is to teach people what they need to know."-carl rogers

In our daily life, it is seen that a number of people arrive at a railway ticket window. If the arrivals too frequent they have to wait for getting railway tickets. In this situation they have to form a queue, called the waiting line, arrivals are called the customers and the persons issuing the tickets are called a server.

Examples of waiting lines

Jobs waiting for processing by a computer, calls arriving at telephone switch board, patients arriving to the doctor's clinic, waiting customers in a queue for getting ticket near the window at a cinema hall, waiting cars in a motor garage for repairing, arriving vehicles at traffic signal etc. The customers may be persons, vehicles, machines etc.

Queuing system



7.1 THE ELEMENTS OF A QUEUING SYSTEM

1) **The Input or Arrival pattern**: It represents the pattern in which customers arrive and join the system. The inter-arrival time also represents it. Inter arrival time is the time period between two successive arrivals. Arrivals may be separated by equal intervals of time or unequal but known intervals of time. Whose probabilities are known which are called random arrivals.

   The number of customers arriving per unit time is called arrival rate.

   The probability of an arrival in the next interval of time does not depend upon customers already in the system means the arrivals are completely random and it follows the Poisson process with mean equals the average number of arrivals per unit time that is λ.

2) **The service mechanism (Or service pattern)**

   The service mechanism is concerned with service time. The time which is requested for servicing a customer is called service time. It may be constant or may change with the customer but it is assumed that the service time is constant for all customers.

   Since the arrival pattern is assumed to be random, the time required for servicing the customer that is service time is also taken as random. Hence the service time follows exponential distribution with mean i.e., reciprocal of the mean rate of service.

3) **The queue discipline**: The simplest discipline is "first come, first served". The customers are served in the order of their arrivals. For example in a Cinema hall, near the ticket windows or at railway stations etc. Other discipline may be "last come, first served" For example in a big Godown, the items which come last are taken out first. Other queue discipline is "service in random order" Notations to describe the nature of service discipline

   FIFO $\rightarrow$ First In, First out

   or FCFS $\rightarrow$ First come, First served

   LIFO $\rightarrow$ Last In, First out

   or FILO $\rightarrow$ First In, Last out

   SIRO $\rightarrow$ Service in Random order

4) **Service arrangements**

   Arrivals directly enter the service station, to get required service, without waiting in the queue. If the server is free at that time. Otherwise they have to wait in the queue till the server gets free.

   Service facilities are of the following types

1. **Single queue – one server**, i.e., the number of queue is one and the number of server is also one.
2. **Single queue – several servers** i.e., the number of queue is one but the number of servers may be more than one. This is an example of parallel counter for providing service.
3. **Several queue – one server** i.e., the number of queue may be more than one but the number of server is one for providing service.
4. **Several servers** i.e., the number of servers are many to provide service. They may be arranged in parallel or in series.

   Parallel means a number of service channels providing identical service facilities. Arrivals have to wait in a single queue until one of the service channels is ready to serve. Series means a customer must pass through all the service channels in sequence before service is completed.

   The service provided in multistage in sequential order, this type of system is known as queue in tandem.

5) **Customer behavior**

Bulk arrival – Generally the customers arrive to the server one by one. But some times customers may arrive in groups. Such arrival is called as bulk arrival.

Jockeying – If there is more than one queue, the customers from one queue will be tempted to join another queue because of its smaller size. This type of behavior of the customers is known as queue jockeying.

Balking – some times customers may leave the queue because the queue is too long and they have no time to wait. This property is known as balking of customers.

Reneging – When a waiting customer leaves the queue due to impatience.

7.2 A LIST OF SYMBOLS

$n \rightarrow$ Number of customers in the system

$c \rightarrow$ Number of servers in the system.

$P_n(t) \rightarrow$ Probability of having n customers in the queuing system at time t.

$P_n \rightarrow$ Steady – state probability of having n customers in the queuing system.

$E_n \rightarrow$ State in which there are n calling customers in the queuing system.

$\lambda_n \rightarrow$ Mean arrival rate of customers.

(Expected number of arrivals per unit time)

$\mu \rightarrow$ Service rate of the server

$\mu_n \rightarrow$ Mean service rate (Expected number of customers served per unit time)

$S \rightarrow$ Number of parallel service stations

$L_s \rightarrow$ Average number of customers waiting in the system.

$L_q \rightarrow$ Average number of customers waiting in the queue.

$W_q \rightarrow$ Average waiting time of the customers in the queue

$W_s \rightarrow$ Average waiting time of the customers in the system

7.3 TRANSIENT, STEADY STATES AND EXPLOSIVE STATE

A system is said to be in "transient state" when its operating characteristics are dependent upon time. A system is said to be in steady state when the behavior of the system is independent of time t. For a sufficiently large time t ($t \rightarrow \infty$) the probability distributions of arrivals, waiting time and servicing time does not depend on time, becomes independent of the initial state of queue.

Mathematically, in Steady State

Let $P_n(t)$ denote the probability that there are n units in the system at time t, the change of $P_n(t)$ with respect to t is given by

$$\frac{dP_n(t)}{dt}$$

$\underset{t\to\infty}{Lim}\, P_n(t) = P_n$ (Independent of t)

$$\Rightarrow \underset{t\to\infty}{Lim} \frac{dP_n(t)}{dt} = \frac{dP_n}{dt} = 0$$

In some situations, if the arrival rate of the system is larger than its service rate, means the queue length will increase with time; such case is called the explosive state.

7.4 PROBABILITY DISTRIBUTIONS IN QUEUING SYSTEMS

It is assumed that customers joining the queuing system arrive in a random manner and follow a Poisson distribution and the intervals between successive arrivals are distributed negative exponentially. Service time are also assumed to be exponentially distributed.

If the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time-interval follows a Poisson distribution. To derive the arrival distribution in queues, we make the following three assumption (axioms)

1. Assuming that there are n units in the system at time t and the probability that exactly one arrival (birth) will occur during small time interval Δt is given by $\lambda\Delta t + O(\Delta t)$. Thus $P_1(\Delta t) = \lambda\Delta t + O(\Delta t)$ where λ is independent of t, is the arrival rate and $O(\Delta t)$ includes the terms of higher order of Δt.
2. The time Δt to be so small that the probability of more than one arrival in time Δt is $[O(\Delta t)]$ almost zero.

OR

The probability of two or more arrivals during the small time interval Δt is negligible

i.e., $$P_0(\Delta t) + P_1(\Delta t) + O(\Delta t) = 1$$

3. The number of arrivals in non-over lapping intervals are statistically independent, that is the process has independent increments.

7.5 DISTRIBUTION OF ARRIVALS (PURE BIRTH PROCESS)

The model in which only arrivals are counted and no departure take place is called pure birth model. The term 'birth' refers to the arrival in the system and the death refers to the departure of a served unit.

We prove the number of arrivals in a time interval (o, t) is a random variable which follows a Poisson distribution with parameter λt.

We want to determine the probability of n arrivals in a time interval of length t, denoted by $P_n(t)$, where $n \geq 0$ is an integer,

When $n > 0$

There may be two mutually exclusive ways of having n units at time $(t + \Delta t)$

1. There are n units in the system at time t and no arrival takes place during the time interval Δt. Hence there will be n units at time $(t + \Delta t)$ also.

   The probability of these two combined events will be

   = probability of n units at time × probability of no arrival during Δt

   $= P_n(t).\ (1 - \lambda\Delta t)$

   [Since probability of exactly one arrival in $\Delta t = \lambda\Delta t$, probability of no arrival is $(1 - \lambda\Delta t)$]

2. There are (n-1) units in the system at time t and one arrival takes place during time Δt. Hence there will be n units in the system at time $(t + \Delta t)$.

   Hence there will be n units in the system at time $(t + \Delta t)$.

The probability of these two combined events will be

$$= P_{n-1}(t) \times \lambda \Delta t$$

[Since probability of (n – 1) units at time t = $P_{n-1}(t)$

Probability of one arrival in time $\Delta t = \lambda\Delta t$]

The probability of n arrivals at time $(t + \Delta t)$,

i.e., $P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t$ ....(A)

When n = 0 i.e., there is no customer in the system. Then

$P_0(t+\Delta t) =$ Probability [no unit at time t) x Probability [no arrival in time Δt]

$$= P_0(t)\ (1-\lambda\Delta t) \qquad \text{............(B)}$$

Thus

$$P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\ \lambda\Delta t \qquad \text{............(A)}$$

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) \qquad \text{............(B)}$$

Rewriting (A) and (B) and dividing throughout by Δt,

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

Taking limit $\Delta t \to 0$ on both side of the above equations

$$\lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t) \qquad \text{............(C)}$$

And

$$\lim_{\Delta t\to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) \qquad \text{............(D)}$$

By the definition of first derivative

$$\lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \frac{dP_n(t)}{dt} = P_n'(t)$$

Equation © implies

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \qquad \text{............(E)}$$

Equation (D) implies

$$P_0'(t) = -\lambda P_0(t) \qquad \text{............(F)}$$

Equation (E) and (F) are called differential difference equations.

Now by the equation (F)

$$\frac{P_0'(t)}{P_0(t)} = -\lambda$$

Integrating both sides with respect to t, we get

$$\log P_0(t) = -\lambda t + A \qquad \text{.............(G)}$$

The constant of integration can be calculated by using the boundary conditions

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n > 0 \end{cases}$$

Substituting n = 0 in equation (G)

$P_0(0) = 1$, gives A = 0

Thus

$$\log P_0(t) = -\lambda t$$

Or $$P_0(t) = e^{-\lambda t}$$

On substituting n = 1, gives $P_1(0) = 0$

By the equation E

$$P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$$

Or $$P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$$

Which is the linear differential equation of first order, can be solved by multiplying both sides of this equation by the integrating factor $e^{\int \lambda dt} = e^{\lambda t}$

$$e^{\lambda t}[P_1'(t) + \lambda P_1(t)] = \lambda$$

$$\frac{d}{dt}[e^{\lambda t}.P_1(t)] = \lambda$$

Integrating both sides w.r.to t,

$$e^{\lambda t} P_1(t) = \lambda t + B \qquad \text{.............(H)}$$

Where B is a constant of integration on putting t = 0

$$e^0 P_1(0) = B$$

$$\Rightarrow \qquad e^0 \times 0 = B$$

$$\Rightarrow \qquad B = 0 \qquad \qquad \text{............(I)}$$

On substituting B = 0 in (H)

$$e^{\lambda t} P_1(t) = \lambda t$$

$$\Rightarrow \qquad P_1(t) = \frac{\lambda t}{e^{\lambda t}} = \frac{\lambda t e^{-\lambda t}}{\lfloor 1}$$

Similarly , Putting n = 2 in (E) we get

$$P_2'(t) + \lambda P_2(t) = \lambda P_1(t)$$

$$P_2'(t) + \lambda P_2(t) = \lambda \left[\frac{\lambda t . e^{-\lambda t}}{\lfloor 1} \right]$$

Or
$$\frac{d}{dt}[e^{\lambda t} . P_2(t)] = \frac{\lambda(\lambda t)}{\lfloor 1}$$

Integrating w.r.to ‘t’

$$e^{\lambda t} P_2(t) = \frac{(\lambda t)^2}{\lfloor 2} + C$$

On putting t = 0

$$P_2(0) = 0 + C$$

Or $\qquad C = 0$

$$n = 2, \; P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{\lfloor 2}$$

Similarly

$$n = 3, \; P_3(t) = \frac{(\lambda t)^3 e^{-\lambda t}}{\lfloor 3}$$

In general

$$n = m, \; P_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{\lfloor m}$$

By mathematical induction, assume n = m the result is true, we prove for n = m + 1 the result is true then by the equation (E)

$$P_{m+1}'(t) + \lambda P_{m+1}(t) = \frac{\lambda (\lambda t)^m e^{-\lambda t}}{\lfloor m}$$

$$\frac{d}{dt}[e^{\lambda t} P_{m+1}(t)] = \frac{\lambda (\lambda t)^m}{\lfloor m}$$

Integratii

Again, pı

⇒

Therefore

Hence, in

Which is

7.6 DISTRIB

Assume tl
no arrival
time mear
from the s

1. Probab

2. Probab
i.e., the
in time

3. The nı
indepe
N(t) ha

First we ol
ways.

Case I :

Case II :

Integrating both sides

$$e^{\lambda t} P_{m+1}(t) = \frac{(\lambda t)^{m+1}}{(m+1)|\underline{m}} + D$$

Again, putting t = 0

$$P_{m+1}(0) = 0 + D$$

$\Rightarrow$ $\qquad 0 = D$

Therefore

$$P_{m+1}(t) = \frac{(\lambda t)^{m+1} e^{-\lambda t}}{|\underline{m+1}}$$

Hence, in general

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{|\underline{n}}$$

Which is Poisson distribution formula with mean λt.

7.6 DISTRIBUTION OF DEPARTURES (OR PURE DEATH PROCESS)

Assume that there are N customers in the system at time t = 0. Also assume no arrival (birth) can occur in the system. Departures occur at a rate μ per unit time means output rate is μ. We want to derive the distribution of departures from the system by assuming following three axioms.

1. Probability (one departure during Δt)] = $\mu\Delta t + O(\Delta t)^2 = \mu\Delta t$

   [Since $O(\Delta t)^2$ is negligible]

2. Probability (more than one departure during Δt] = $O(\Delta t)^2 = O$

   i.e., the term Δt is so small that the probability of more than one departure in time Δt is negligible.

3. The number of departures in non-overlapping intervals are statistically independent and identically distributed random variable means the process N(t) has independent increments.

First we obtain the differential difference equation in three mutually exclusive ways.

Case I : When $O < n < N$

As in the pure birth process, we have

$$P_n(t + \Delta t) = P_n(t)[1 - \mu\Delta t] + P_{n+1}(t)\mu\,\Delta t \qquad \text{............(A)}$$

Case II : When n = N

Since there are exactly N units in the system, $P_{n+1}(t) = 0$

$$P_N(t+\Delta t) = P_N(t)[1-\mu\Delta t] \qquad \text{............(B)}$$

Case III : When n = 0

$$P_0(t+\Delta t) = P_0(t) + P_1(t)\mu\Delta t] \qquad \text{............(C)}$$

Since there is no unit in the system at time t, the question of any departure Δt does not arise. Therefore in this case probability of no departure is unity.

Dividing by Δt and taking limit Δt → 0

In equation (A), (B) & (C) we get

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\mu P_n(t) + \mu P_{n+1}(t) \qquad \text{- from (A) } 0 < n < N$$

$$\lim_{\Delta t \to 0} \frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} = -\mu P_N(t) \qquad \text{- from (B) } n = N$$

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = \mu P_1(t) \qquad \text{-from (C) } n = 0$$

Or
$$P_n'(t) = -\mu P_n(t) + \mu P_{n+1}(t) \qquad \text{............(D)}$$

$$P_N'(t) = -\mu P_N(t) \qquad \text{............(E)}$$

$$P_0'(t) = \mu P_1(t) \qquad \text{............(F)}$$

from equation (E), we get $\dfrac{P_N'(t)}{P_N(t)} = -\mu$

or
$$\frac{d}{dt}[\log P_N(t)] = -\mu$$

Integrating w.r.to t on both sides,

$$\log P_N(t) = -\mu t + A$$

Where A is the constant of integration. Its value can be determined using the boundary conditions

$P_N(0) = 1$ thus A = 0

$$\log P_N(t) = -\mu t$$

or
$$P_N(t) = e^{-\mu t}$$

In the equation (D), put n = N-1, then

$$P'_{N-1}(t) = -\mu\, P_{N-1}(t) + \mu\, P_N(t)$$

$$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu e^{-\mu t}$$

$$P'_{N-1}(t) + \mu P_{N-1}(t) = \mu e^{-\mu t}$$

The solution of this equation is given by

$$P_{N-1}(t).e^{\int \mu\, dt} = \int \mu\, e^{-\mu t}\,(e^{\int \mu\, dt})\; dt + C$$

$$P_{N-1}(t).e^{\mu t} = \int \mu\; dt\; + C$$

Or $\quad P_{N-1}(t) = \mu t.e^{-\mu t} + C\, e^{-\mu t}$

Put $\quad$ t = 0, to determine C,

$P_{N-1}(0) = 0$ gives C = 0

Hence $\quad P_{N-1}(t) = \dfrac{\mu t.e^{-\mu t}}{\underline{|1}}$

Now putting n = N – 2 in equation (D), we get

$$P_{N-2}(t) = \frac{(\mu t)^2\, e^{-\mu t}}{\underline{|2}}$$

For n = N – i, we get,

$$P_n(t) = P_{N-i}(t) = \frac{e^{-\mu t}\,(\mu t)^i}{\underline{|i}}$$

$$= \frac{e^{-\mu t}\,(\mu t)^{N-n}}{\underline{|N-n}} \qquad \text{for } n = 1, 2, \ldots\ldots N.$$

We have $\displaystyle\sum_{n=0}^{N} P_n(t) = 1$

Or $\quad \displaystyle 1 = \sum_{n=0}^{N} P_n(t)$

$$= P_0(t) + \sum_{n=1}^{N} P_n(t)$$

$$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t)$$

$$= 1 - \sum_{n=1}^{N} \frac{(\mu t)^{N-n}\, e^{-\mu t}}{\lfloor N-n}$$

$$P_n(t) = \begin{cases} \dfrac{(\mu t)^{N-n}\, e^{-\mu t}}{\lfloor N-n} & for \quad n = 1, 2, \ldots\ldots N \\ 1 - \displaystyle\sum_{n=1}^{N} \frac{(\mu t)^{N-n}\, e^{-\mu t}}{\lfloor N-n} & for \quad n = 0 \end{cases}$$

The number of departures in time t follows the truncated Poisson distribution.

7.7 MODEL I

(M/M/1) : (GD/∞/∞)

It represents a single – server exponential queuing system with Poisson or Markovian arrivals, Poisson or Markovian departures. The inter – arrival times (time between successive arrivals) are independent exponential random variables having mean 1/λ. The successive service times are also assumed to be independent exponential random variables having mean 1/μ. This model is based on 'First in – First out'.

1 denotes the single server. In this exponential model GD denotes queue discipline is general discipline. First ∞ denotes number of customers permitted in the system are infinite. Second infinite (∞) denotes the size of source from which customers arrive is infinite.

1. Obtain the difference differential equations

Let the probability that there are n customers in the system at time t is $P_n(t)$.

Then the probability that the system has n – customers at time (t + Δt) may be expressed as the sum of the combined probability of following four mutually exclusive and exhaustive events.

$$P_n(t+\Delta t) = P_n(t) \times probability(no\ arrival\ in\ \Delta t) \times probability(no\ service\ completed\ in\ \Delta t)$$
$$+ P_n(t) \times probability\ (one\ arrival\ in\ \Delta t) \times probability(one\ service\ completed\ in\ \Delta t)$$
$$+ P_{n+1}(t) \times probability\ (on\ arrival\ in\ \Delta t) \times probability(one\ service\ completed\ in\ \Delta t)$$
$$+ P_{n-1}(t) \times probabili..\ (one\ arrival\ in\ \Delta t) \times probability(no\ service\ completed\ in\ \Delta t)$$
$$= P_n(t)(1-\lambda\Delta t)(1-\mu\Delta t) + P_n(t)(\lambda\Delta t)\ (\mu\Delta t)$$
$$+ P_{n+1}(t)\mu\Delta t(1-\lambda\Delta t) + P_{n-1}(t)\ \lambda\Delta t\,(1-\mu\Delta t)$$

$$P_n(t+\Delta t) - P_n(t) = -(\lambda+\mu)P_n(t)\,\Delta t + P_{n+1}(t)\,\mu\Delta t + P_{n-1}(t)\,\lambda\Delta t$$

Dividing by Δt and taking limit Δt → 0, we get

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu)\,P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t)$$

$n \geq 1$

Or $$P_n'(t) = -(\lambda+\mu)P_n(t) + \mu P_{n+1}(t) + P_{n-1}(t) \qquad \text{............(A)}$$

If there is no customer in the system at time (t + Δt), there will no service during Δt. then for n = 0.

$$P_0(t+\Delta t) = P_0(t) \times probability(no\ arrival\ in\ \Delta t) + P_1(t) \times probability(no\ arrival\ in\ \Delta t)$$
$$\times\ probability(one\ service\ completed\ in\ \Delta t)$$
$$= P_0(t)(1-\lambda\Delta t) + P_1(t)(1-\lambda\Delta t)\mu\Delta t$$

$$P_0(t+\Delta t) - P_0(t) = -\lambda P_0(t)\Delta t + P_1(t)(1-\lambda\Delta t)\mu\Delta t$$

Dividing by Δt and taking limit $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t)\ ,\ n = 0$$

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad \text{.......(B)}$$

Equation (A) & (B) are called differential difference equations.

$$P_n'(t) = -(\lambda+\mu)P_n(t) + P_{n+1}(t)\mu + P_{n-1}(t)\lambda \quad n \geq 1$$

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad n = 0$$

2. Obtain the steady – state difference differential equations:

The probability of n customers in the system at any point of time after steady state is reached, taking the limit as $t \to \infty$ in (A) & (B)

$\frac{\lambda}{\mu} < 1$ as $t \to \infty$

If $\lambda = \mu$ there is no queue

If $\frac{\lambda}{\mu} > 1$ the state is called the explosive state.

In the steady state

$P_n(t) \to P_n$ and $\frac{d}{dt} P_n(t) \to 0$ are $t \to \infty$

From (A) and (B)

$$O = -(\lambda + \mu)P_n + \mu P_{n+1} + \lambda P_{n-1} \qquad n \geq 1$$

Or $\qquad +(\lambda + \mu)P_n = \mu P_{n+1} + \lambda P_{n-1}$

And $\qquad O = -\lambda P_0 + \mu P_1 \qquad\qquad n = 0$

Or $\qquad \lambda P_0 = \mu P_1$

From the above equations

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{\lambda + \mu}{\mu}\right) P_1 - \frac{\lambda}{\mu} P_0$$

$$= \left(\frac{\lambda + \mu}{\mu}\right) \frac{\lambda}{\mu} P_0 - \frac{\lambda}{\mu} P_0$$

$$= \left(\frac{\lambda}{\mu} + 1\right) \frac{\lambda}{\mu} P_0 - \frac{\lambda}{\mu} P_0$$

$$= \frac{\lambda}{\mu} P_0 \left[\left(\frac{\lambda}{\mu} + 1\right) - 1\right]$$

$$= \frac{\lambda}{\mu} P_0 \left(\frac{\lambda}{\mu}\right)$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Similarly

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$\vdots$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \qquad n \geq 0$$

3. To determine

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P$$

$$P_0 + \left(\frac{\lambda}{\mu}\right) P$$

$$P_0 \left[\frac{1}{1-\left(\frac{\lambda}{\mu}\right.}\right.$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$P_0 = 1 - \rho$$

Which is the

4. Probability th

Or

Which is the there are n cu

5. Probability th

6. Average or

3. To determine P_0, the probability that there is no customer in the system

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$P_0 + \left(\frac{\lambda}{\mu}\right) P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \dots\dots\dots + \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$P_0 \left[\frac{1}{1-\left(\frac{\lambda}{\mu}\right)}\right] = 1$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$P_0 = 1 - \rho \text{ where } \rho = \frac{\lambda}{\mu} < 1$$

Which is the probability that there is no customer in the system.

4. Probability that there are n customer in the system

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Or $$P_n = \rho^n (1 - \infty)$$

Which is the probability distribution of queue length or the probability that there are n customers in the system.

5. Probability that a service channel is busy or its traffic intensity

$$\rho = \frac{\lambda}{\mu} = \frac{mean\ arrival\ rate}{mean\ service\ rate}$$

6. Average or expected number of customers in the system is

$$L_s = \sum_{n=0}^{\infty} nP_n$$

$$= \sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\left[1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\ldots\ldots\right]$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\left[\frac{1}{\left(1-\frac{\lambda}{\mu}\right)^2}\right]$$

$$= \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\rho}{1-\rho}$$

Or $$L_s = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

7. Average length of waiting line or average or expected number of customers in the queue or average or expected queue length.

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n$$

$$= \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n$$

$$= \sum_{n=0}^{\infty} nP_n - \left[\sum_{n=0}^{\infty} P_n - P_0\right]$$

$$= \frac{\rho}{1-\rho} - [1-(1-\rho)]$$

$Lq =$

8. Average or ex

W_q :

9. Average waiti

W_n :

10. Average waiti

W_s =

W_s =

$P(W>0) = 1$ ·

Therefore W_s =

11. Probability tha

Pr

$$= \frac{\rho}{1-\rho} - \rho$$

$$= \frac{\rho^2}{1-\rho}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^2}{1-\frac{\lambda}{\mu}}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$Lq = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

8. Average or expected waiting time of a customer in the queue.

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$$

9. Average waiting time of a customer who has to wait

$$W_n = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

10. Average waiting time of a customer in the system

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

$$W_s = \frac{L_q}{P(W>0)}$$

$$P(W>0) = 1 - P(W=0) = 1 - P_0 = 1-(1-\rho) = \rho$$

Therefore $W_s = \dfrac{\rho}{\mu(1-\rho)\rho} = \dfrac{1}{\mu-\lambda}$

11. Probability that queue length is greater than or equal to n

$$\text{Probability } (\geq n) = \left(\frac{\lambda}{\mu}\right)^n = \rho^n$$

ted number of

12. Probability density function of both waiting and service time distribution

$$(\mu-\lambda)e^{-(\mu-\lambda)t}$$

13. Probability density function of waiting time distribution

$$\frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)t} \qquad \text{for } t>0$$

$$\lambda\left(1-\frac{\lambda}{\mu}\right) \qquad \text{for } t=0$$

7.8 MODEL II

(M/M/1): (GD/N/∞)

In this model, 1 denotes the single server. GD denotes queue discipline is general discipline. N denotes that maximum number of customers permitted in the system is N. Infinite (∞) denotes the size of source from which customers arrive is finite.

This model is different from model I, in the model II number of customers permitted in the system is N whereas in the model I number of customers permitted in the system is infinite. Therefore the difference equations of model I are valid for this model as long as $n < N$.

The additional difference equation for $n = N$, is

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t] + P_1(t)\mu\Delta t + O(\Delta t)$$

$$\text{for } n=0 \quad \ldots\ldots\ldots \text{(A)}$$

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda+\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + O(\Delta t)$$

$$\text{for } n = 1, 2, \ldots\ldots\ldots N-1 \ldots\ldots\ldots \text{(B)}$$

and

$$P_N(t+\Delta t) = P_N(t)[1-(0+\mu)\Delta t] + P_{N-1}(t)\lambda\Delta t + O\times\mu\Delta t + O(\Delta t)$$

$$P_N(t+\Delta t) = P_N(t)[1-\mu\Delta t] + P_{N-1}(t)\lambda\Delta t + O(\Delta t) \ldots\ldots\ldots \text{(C)}$$

for $n = N$, $P_{N+1}(t) = 0$, $\lambda = 0$

Now dividing equation (A), (B) and (C) by Δt and taking limit as $\Delta t \to 0$

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad \text{for } n=0$$

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) \qquad \text{for n = 1, 2, ...N-1}$$

$$P_N'(t) = -\mu P_N(t) + \lambda P_{N-1}(t) \qquad \text{for n = N}$$

In the case of steady state, when $t \to \infty$ $P_n(t) \to P_n$ and hence $P_n'(t) \to 0$ then the steady state difference equations are given by

$$O = -\lambda P_0 + \mu P_1 \qquad \text{for n = 0}$$

$$O = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1} \qquad \text{for n = 1, 2, ...N-1}$$

$$O = -\mu P_n + \lambda P_{N-1} \qquad \text{for n = N}$$

Form the above equation

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Similarly

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\vdots$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \qquad n < N$$

$$P_N = \left(\frac{\lambda}{\mu}\right)^N P_0 \qquad n = N$$

For obtaining the value of P_0, we use the boundary conditions

$$\sum_{n=0}^{N} P_n = 1$$

$$P_0\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \ldots\ldots\ldots\ldots\ldots + \left(\frac{\lambda}{\mu}\right)^N\right] = 1$$

Or $$P_0\left[\frac{1-\rho^{N+1}}{1-\rho}\right]=1 \qquad \text{where } \rho=\frac{\lambda}{\mu}$$

Or $$P_0=\frac{1-\rho}{1-\rho^{N+1}}$$

$$1=\sum_{n=0}^{N}P_n=\sum_{n=0}^{N}\rho^n\ P_0$$

$$=P_0\sum_{n=0}^{N}\rho^n$$

$$=\begin{cases}P_0\left[\dfrac{1-\rho^{N+1}}{1-\rho}\right] & \rho\neq 1\\ P_0(N+1) & \rho=1\end{cases}$$

Or $$P_0=\begin{cases}\dfrac{1-\rho}{1-\rho^{N+1}} & \rho\neq 1\\ \dfrac{1}{N+1} & \rho=1\end{cases}$$

Hence $$P_n=\begin{cases}\dfrac{(1-\rho)\rho^n}{1-\rho^{N+1}} & \rho\neq 1\\ \dfrac{1}{N+1} & \rho=1\end{cases} \qquad 0\leq n\leq N$$

It $\lambda<\mu$ and $N\to\infty$ we get model I.

- Average number of customers in the system is given by

$$L_s=\sum_{n=0}^{N}nP_n$$

$$=\sum_{n=0}^{N}nP_0\rho^n$$

$$=P_0\,\rho\sum_{n-0}^{N}n\,\rho^{n-1}$$

$$=P_0\,\rho\sum_{n=0}^{N}\frac{d}{d\rho}\rho^n$$

- Effect
- Expec
- Expec
- Expec

MODEL I :(M/M

Example 7.1. Tl
Poisson distributic
also follows Poiss
a) The pı
i.e., P
b) The pı
i.e., P
c) Avera
i.e., L
d) Avera

$$= P_0 \rho \frac{d}{d\rho} \sum_{n=0}^{N} \rho^n$$

$$= P_0 \rho \frac{d}{d\rho} \left(\frac{1-\rho^{N+1}}{1-\rho} \right)$$

$$= \begin{cases} \dfrac{\rho}{1-\rho^{N+1}} \left[\dfrac{1-(N+1)\rho^N + N\rho^{N+1}}{1-\rho} \right] & \rho \neq 1 \\ \dfrac{N}{2} & \rho = 1 \end{cases}$$

- Effective arrival rate $\lambda^1 = \lambda(1 - P_N)$
- Expected number of customers waiting in the queue

$$L_q = L_s - \frac{\lambda^1}{\mu}$$

- Expected waiting time of a customer in the system

$$W_s = \frac{L_s}{\lambda^1}$$

- Expected waiting time of a customer in the queue

$$W_q = W_s - \frac{1}{\mu}$$

SOLVED EXAMPLES

MODEL I :(M/M/1) : (GD/∞/∞)

Example 7.1. The arrival rate of customers at a railway ticket window follows Poisson distribution with a mean of 50 per hour. The service rate of a window clerk also follows Poisson distribution with a mean of 70 per hour. Find.

a) The probability of having 0 customer in the system.
   i.e., P_0

b) The probability of having 5 customers in the system
   i.e., P_5

c) Average number of waiting customers in the system
   i.e., L_s

d) Average number of waiting customers in the queue

i.e., L_q

e) Average waiting time of customer in the system

i.e., W_s

f) Average waiting time of customer in the queue

i.e., W_q

Solution Given that

Arrival rate, $\lambda = 50$ per hour

Service rate, $\mu = 70$ per hour

The utilization factor $\rho = \dfrac{\lambda}{\mu} = \dfrac{50}{70} = 0.71$

a) The probability of having 0 customer in the system

$$P_0 = 1 - \rho = 1 - 0.71 = 0.29$$

Ans.

b) The probability of having 5 customer in the system

$$\begin{aligned} P_5 &= (1-\rho)\rho^5 \\ &= (1-0.71)(0.71)^5 \\ &= (0.29)(0.180) \\ &= 0.0523 \end{aligned}$$

Ans.

c) Average number of waiting customer in the system

$$L_s = \frac{\rho}{1-\rho} = \frac{0.71}{1-0.71} = \frac{0.71}{0.29} = 2.448$$

d) Average number of waiting customer in the queue

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.71)^2}{(1-0.71)} = \frac{0.5041}{0.29} = 1.7382$$

e) Average waiting time of customer in the system

$$W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{70(1-0.71)} = \frac{1}{70(0.29)} = \frac{1}{20.3} = 0.049$$

f) Average waiting time of customer in the queue

$$W_q = \frac{\rho}{\mu(1-\rho)} = \frac{0.71}{70(1-0.71)} = \frac{0.71}{70 \times 0.29} = 0.0349$$

Example 7.2.
an exponential c
which they cam
average rate of
Poisson distribu
for the mechani
brought in?

Solution Given

$\lambda =$

$\mu =$

$\rho = \dfrac{\lambda}{\mu} =$

The proba

$P_0 =$

Average ic

Average n

Example 7.3.
can serve 12 stu
Poisson distribu
find.

1. Ave
2. Ave
3. Ave
4. Ave

Solution

Give

Arri

Serv

Example 7.2. A motor garage mechanic finds that the time spent on his jobs has an exponential distribution with mean 40 minutes. If he repairs cars in the order in which they came in and if the arrival of cars is approximately Poisson with an average rate of 8 per 8 hours day and the service rate of the mechanic follows Poisson distribution with mean of 15 cars per day. What is the expected idle time for the mechanic for each day? How many jobs are ahead of the average car just brought in?

Solution Given that

$$\lambda = 8 \quad \text{cars per day}$$

$$\mu = 15 \quad \text{cars per day}$$

$$\rho = \frac{\lambda}{\mu} = \frac{8}{15} = 0.533$$

The probability for the mechanic to be idle is

$$P_0 = 1 - \rho = 1 - 0.533 = 0.467$$

Average idle time per day $= 8 \times 0.467$

$$= 3.736 \text{ hours}$$

Average number of cars in the system

$$= \frac{\rho}{1-\rho} = \frac{0.533}{1-0.533} = \frac{0.533}{0.467}$$

$$= 1.141 \text{ cars}$$

Example 7.3. On an average of 10 students arrive every 5 minutes while the clerk can serve 12 students in 5 minutes in an engineering college to pay fees. Assuming Poisson distribution for arrival rate and exponential distribution for service rate find.

1. Average number of students in queue
2. Average number of students in the system
3. Average time a student waits before being served
4. Average time a student spends in the system.

Solution

0.049

Given that

Arrival rate $\lambda = \frac{10}{5} = 2$ students / minute

Service rate $\mu = \frac{12}{5} = 2.5$ students / minute

1. Average number of students in queue

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(2)^2}{2.4(2.4-2)} = \frac{4}{(2.4)(0.4)}$$

$$= 4.166$$

2. Average number of students in the system

$$= \frac{\lambda}{\mu-\lambda} = \frac{2}{2.4-2} = \frac{2}{0.4} = 5$$

3. Average time a student waits before served

$$= \frac{\lambda}{\mu(\mu-\lambda)} = \frac{2}{2.4(2.4-2)} = \frac{2}{(2.4)(0.4)}$$

4. Average time a student spends in the system

$$= \frac{1}{\mu-\lambda} = \frac{1}{2.4-2} = \frac{1}{0.4} = 2.5 \text{ minutes}$$

Example 7.4. The service time follows exponential (negative) distribution and the service rate follows Poisson distribution at a single window to update account book in a bank. The arrival rate of customers follows Poisson distribution. The arrival rate and the service rate are 35 customers per hour and 45 customers per hour respectively. Find

1. Utilization of the updating clerk
2. Average number of waiting customers in the queue
3. Average number of waiting customers in the system
4. Average waiting time per customers in the queue
5. Average waiting time per customers in the system

Solution : Given that

The arrival rate of customers at a single window in a bank to update account book

$\lambda = 35$ customers per hour

And the service rate $\mu = 45$ customers per hour

1. Utilization of the updating clerk $\rho = \frac{\lambda}{\mu} = \frac{35}{45}$

$$= 0.777$$

2. Average number of waiting customers in the queue

3. Av

4. Av

5. Av

Example 7.5.
authority (RTA
wheeler is regis

1. The
2. The
3. The
20

Solution : Give

Registrati

1. The
offi

2. The

We

$$L_q = \frac{\rho^2}{(1-\rho)} = \frac{(0.777)^2}{(1-0.777)} = \frac{0.6048}{0.2223}$$

$$= 2.72064 \text{ customers}$$

3. Average number of waiting customer in the system

$$L_s = \frac{\rho}{1-\rho} = \frac{0.777}{1-0.777} = \frac{0.777}{0.223} = 3.48 \text{ customers}$$

4. Average waiting time per customer in the queue

$$W_q = \frac{\rho}{\mu-\lambda} = \frac{0.777}{45-35} = \frac{0.777}{10} = 0.0777 \text{ hour}$$

5. Average waiting time per customer in the system

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{45-35} = \frac{1}{10} = 0.1 \text{ hour}$$

Example 7.5. Assuming that the time between registrations at a road transport authority (RTA) office follows an exponential distribution where one new two wheeler is registered after every 10 minutes find

1. The average number of registrations of two wheeler per year
2. The probability that no registration will occur in any one day.
3. The probability of 40 registrations by the end of next 3 hours given that 20 registrations were done during the last 2 hours.

Solution : Given that

One new two wheeler is registered after every 10 minutes

$$\text{Registration rate per day } \lambda = \frac{24\times 60}{10} = 144$$

1. The average number of registrations of two wheeler per year at RTA office

$$\lambda t = 144 \times 365 \text{ days}$$

$$= 52560 \text{ registrations / year}$$

2. The probability that no registration will occur in any one day.

We have $P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{\lfloor n}$

$\lambda = 144,\ n = 0,\ t = 1$ day

$$P_0(1) = \frac{(144\times 1)^0\, e^{-144\times 1}}{\lfloor 0}$$

$$= e^{-144}$$

$$= 2.88\times 10^{-63}$$

3. The probability of 40 registrations by the end of next 3 hours given that 20 registrations were
   done during the last two hours.

$$\lambda = \frac{60}{10} = 6 \text{ registrations / hour}$$

$$P_n(t) = \frac{(\lambda t)^n\, e^{-\lambda t}}{\lfloor n}$$

$$n = 20 \text{ registrations}$$

Since in the last two hours 20 registrations has been done and we have to find probability of 40 registrations by the end of next 3 hours

$\Rightarrow t = 1$

$$P_{20}(1) = \frac{(6\times 1)^{20}\, e^{-6\times 1}}{\lfloor 20}$$

$$= \frac{(6)^{20}\, e^{-6}}{\lfloor 20}$$

$$= \frac{(3.6561\times 10^{15})(2.4787\times 10^{-3})}{(2.4329\times 10^{18})}$$

$$= 3.7249\times 10^{-6}$$

Example 7.6. The arrival rate of customers at a railway ticket single booking counter follows Poisson distribution and the service time for the customers follows exponential (negative) distribution and hence service rate follows Poisson distribution the arrival rate and the service rate are 8 customers per hour and 14 customers per hour, respectively. Find

1. Utilization of the booking clerk
2. Average number of waiting customers in the queue
3. Average number of waiting customers in the system
4. Average waiting time per customer in the queue
5. Average waiting time per customer in the system

Solution

1. Uti
2. Av
3. Av
4. Av
5. Av

Example 7.7.
30 customers p
cashier is 34 pe
apply. Find

1. Wh
2. Ave
3. Ave
4. Ave
5. Ave

Solution : Give

Solution : Given that

The arrival rate of the customers $\lambda = 8$ customer/hour

The service rate $\mu = 14$ customers / hour

1. Utilization of the booking clerk $\rho = \frac{\lambda}{\mu} = \frac{8}{14} = 0.5714$

2. Average number of waiting customers in the queue

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.5714)^2}{(1-0.5714)} = 0.761 \text{ customers}$$

3. Average number of waiting customers in the system

$$L_s = \frac{\rho}{1-\rho} = \frac{0.5714}{1-0.5714} = 1.333 \text{ customers}$$

4. Average waiting time per customer in the queue

$$W_q = \frac{\rho}{(\mu-\lambda)} = \frac{0.5714}{14-8} = 0.095 \text{ hours}$$

5. Average waiting time per customer in the system

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{14-8} = 0.1666 \text{ hours}$$

Example 7.7. A shopping mall has a single cashier. Arrival rate of customers is 30 customers per hour. The average number of customers that can be serviced by cashier is 34 per hour. If the conditions for the use of single channel queuing model apply. Find

1. What is the probability that the cashier is idle?
2. Average number of customers in the queuing system?
3. Average number of customer in the queue?
4. Average time a customer spends in the queue waiting for service.
5. Average time a customer spends in the system?

Solution : Given that

Mean arrival rate $\lambda = 30$ customers per hour

Mean service rate $\mu = 34$ customers per hour

$$\rho = \frac{\lambda}{\mu} = \frac{30}{34} = 0.88235$$

1. Probability that the cashier is idle

$$P_0 = 1-\rho$$
$$= 1-0.88235$$
$$= 0.1176$$

2. Average number of customers in the queuing, system

$$L_s = \frac{\rho}{1-\rho} = \frac{0.88235}{1-0.88235} = \frac{0.88235}{0.11765}$$
$$= 7.499 \text{ customers}$$

3. Average number of customers in the queue

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.88235)^2}{1-0.88235} = \frac{0.77854}{0.1176}$$
$$= 6.62025 \text{ customers}$$

4. Average time a customer spends in the queue waiting for service

$$W_q = \frac{\rho}{\mu-\lambda} = \frac{0.88235}{34-30}$$
$$= 0.22058 \text{ minutes}$$

5. Average time a customer spend in the system

$$W_S = 1/\mu\text{-}\lambda = 1/34-30 = 0.25 \text{ minutes}$$

Example 7.8. A car mechanic repairs cars in the order in which cars come and if the arrival of cars is approximately Poisson with an average rate of 10 per 8 hour day, and the car mechanic finds that the time taken for his work has an exponential distribution with mean 30 minutes. What is his expected idle time each day and the average number of cars waiting in the system for repairing

Solution : Given that

Mean arrival rate $\lambda = 10$ cars per day

Mean service rate $\mu = \frac{8}{30} \times 60 = 16\,cars/day$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{16} = 0.625$$

1. Mechanic expected idle time each day

$$P(0) = 1-\rho = 1-0.625 = 0.375$$

2. Average number of cars waiting in the system for repairing

$$L_s = \frac{\rho}{1-\rho} = \frac{0.625}{1-0.625} = 1.666 \text{ cars}$$

Example 7.9.
counter clerk tal
deposit counter i
bank manager re
customers proble
1. Wha
2. Wha
to wa
3. What

Solution Given tl
T
T

1. the pc

2. The a
to dep

3. What

Example 7.10. In
average interval o
minutes to serve a
1. Averag
2. Expect

Example 7.9. A bank has one counter who takes all customers deposit. One counter clerk takes, on an average, 4 minutes per customer. Customers come to deposit counter in a random manner on an average of 10 customers per hour. The bank manager received a large number of customers complaint for solving the customers problem, he decided to find answer of following questions.

1. What portion of his time is the counter clerk expected to be idle?
2. What is the average length of time that a customer would be expected to wait to deposit their amount?
3. What is the average length of the waiting time to be expected?

Solution Given that

The mean arrival rate $\lambda = 10$ customers / hour

The mean service rate $\mu = 1$ customer in 4 minutes

$$= \frac{60}{4} = 15 \text{ customer / hour}$$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{15}$$

1. the portion of his time is the counter clerk expected to be idle

$$P_0 = 1 - \rho$$

$$= 1 - \frac{10}{15} = \frac{1}{3} = 0.33$$

2. The average length of time that a customer would be expected to wait to deposit their amount

$$W_q = \frac{\rho}{\mu - \lambda} = \frac{0.666}{15-10} = \frac{0.666}{5} = 0.133 \text{ hour}$$

3. What is the average length of waiting time to be expected

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.666)^2}{1-0.666} = \frac{0.444}{0.334}$$

$$= 1.329 \approx 1.33$$

Example 7.10. In a big bazaar at a billing counter, the customers arrive at the average interval of six minutes where as the counter clerk takes on an average 5 minutes to serve a customer. Find

1. Average waiting time of the customers at a billing counter
2. Expected average waiting time in the line

3. Average number of customers in the line
4. Average number of customers in the service counter area.
5. Probability that the counter clerk is idle
6. Probability that customer has to wait more than 30 minutes in the system.
7. Probability of finding more than 3 customers in the system.
8. Probability of having 5 customers in the system

Solution : Given that

The mean arrival rate $\lambda = 6$ minutes, one customer

$$= \frac{60}{6} = 10 \text{ customer / hours}$$

The mean service rate $\mu = 5$ minutes, one customer

$$= \frac{60}{5} = 12 \text{ customer / hour}$$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{12} = 0.8333$$

1. Average waiting time of the customers at a billing counter

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12-10} = \frac{1}{2} = 0.5 \text{ hours}$$

2. Expected average waiting time in the line

$$W_q = \frac{\rho}{\mu - \lambda} = \frac{0.833}{12-10} = \frac{0.8303}{2} = 0.4166 \text{ hours}$$

3. Average number of customers in the line

$$L_q = \frac{\rho^2}{1-\rho}$$

$$L_q = \frac{(0.8333)^2}{1-(0.8333)} = \frac{0.6944}{0.1666}$$

$$= 4.1680 \text{ customers}$$

$$\approx 4 \text{ customers}$$

4. Average number of customers in the service counter area

$$L_s = \frac{\rho}{1-\rho} = \frac{0.8333}{1-0.8333} = \frac{0.8333}{0.1666}$$

$$= 5.0018 \text{ customers}$$

$$\approx 5 \text{ customers}$$

5. Prob

6. Prob syste

I

7. Prob

8. Proba

Example 7.11. C and that shop has processed by the c

1. The pı
2. The av
3. The av
4. The av
5. The av

Solution : Given tl

The arrival r

The average

e system.

5. Probability that the counter clerk is idle

$$P_0 = 1 - \rho$$
$$= 1 - 0.8333$$
$$= 0.1666$$

6. Probability that customer has to wait more than 30 minutes in the system

$$P(T > t = 30 \text{ minutes} = \tfrac{1}{2} \text{ hours})$$
$$= e^{-(\mu-\lambda)t}$$
$$= e^{-(12-10)1/2} = e^{-1} = 0.368$$

7. Probability of finding more than 3 customers in the system

$$P(k > 3) = \rho^{k+1}$$
$$= (0.8333)^{3+1}$$
$$= (0.8333)^4$$
$$= 0.482$$

8. Probability of having 5 customers in the system

$$P_{(5)} = \rho^n (1 - \rho)$$
$$= (0.833)^5 (1 - 0.8333)$$
$$= (0.40107)(0.167)$$
$$= 0.06697$$

Example 7.11. Customers arrive at rate of 30 customers per hour in a boot house and that shop has a single cashier. The average number of customers that can be processed by the cashier is 34 per hour. Calculate

1. The probability that the cashier is idle
2. The average number of customers in the queuing system
3. The average number of customers in the queue
4. The average time a customer spends in the system
5. The average time a customer spends in the queue waiting for service.

Solution : Given that

The arrival rate of customers in a boot house is $\lambda = 30$ customers / hour

The average number of customers that can be processed by the casher is

$$\mu = 34 \text{ customers / hour}$$

1. The probability that the cashier is idle

We know that

The utilization factor $\rho = \dfrac{\lambda}{\mu} = \dfrac{30}{34} = 0.882$

Now

$$P_0 = 1 - \rho = 1 - 0.882 = 0.118$$

2. The average number of customers in the queuing system

$$L_s = \frac{\rho}{1-\rho} = \frac{0.882}{1-0.882} = \frac{0.882}{0.118} = 7.474$$

3. The average number of customer in the queue

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.882)^2}{1-0.882} = \frac{0.7779}{0.118} = 6.592$$

4. The average time the customer spends in the system

$$W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{34(1-0.882)} = \frac{1}{34(0.118)} = 0.249 \text{ hours}$$

$$= 14.97 \text{ minutes}$$

5. The average time a customer spends in the queue waiting for service

$$W_q = \frac{\rho}{\mu(1-\rho)} = \frac{0.882}{34(1-0.882)} = \frac{0.882}{34 \times 0.118}$$

$$= 0.2198 \text{ hours}$$

$$\doteq 13.18 \text{ minutes}$$

MODEL II : (M/M/1): (GD/N/∞)

Example 7.12. Assuming that the inter arrival time in a railway marshalling yard follows an exponential distribution, where goods train arrive at the rate of 40 trains per day. Service time is also follows an exponential distribution with mean of 25 minutes. Find

1. Probability that the yard is empty
2. Average queue length, given that, the capacity of the yard is 10 trains

Solution Given that

$$\lambda = 40 \text{ trains / day}$$

$$= \frac{40}{60 \times 24} = \frac{1}{36} \text{ trains / minutes}$$

$$\mu = \frac{1}{25} \text{ trains / minutes}$$

$$\rho = \frac{\lambda}{\mu} = \frac{25}{36} = 0.6944$$

1. Probability that the yard is empty

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} \qquad \text{where N = 10}$$

$$= \frac{1-0.6944}{1-(0.6944)^{10+1}}$$

$$= \frac{0.3055}{1-0.0181} = \frac{0.3055}{0.9818} = 0.3114$$

2. Average queue length

$$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n$$

$$= (0.3114)\sum_{n=0}^{N} n(0.3114)^n$$

$$L_s = (0.3112)[1(0.6944)^1 + 2.(0.6944)^2 + 3(0.6944)^3 + 4(0.6944)^4 + 5(0.6944)^5 + 6(0.6944)^6 + 7(0.6944)^7 + 8(0.6944)^8 + 9(0.6944)^9 + 10(0.6944)^{10}]$$

$$= (0.3112)[0.6944 + 0.9643 + 1.0045 + 0.9300 + 0.8072 + 0.6726 + 0.5449 + 0.4324 + 0.3378 + 0.2606]$$

$$= (0.3112)(6.6487)$$

$$= 2.06$$

Ans.

$$L_s = \frac{\rho}{1-\rho^{N+1}}\left[\frac{1-(N+1)\rho^N + N\rho^{N+1}}{1-\rho}\right]$$

$$= \frac{0.6944}{1-(0.6944)^{11}}\left[\frac{1-(10+1)(0.6944)^{10} + 10(0.6944)^{10+1}}{1-0.6944}\right]$$

$$= \frac{0.6944}{0.9818}\left[\frac{1-0.2867+0.1810}{0.3056}\right]$$

$$= \frac{0.6944 \times 2.9263}{0.9818}$$
$$= 2.0697$$
$$\approx 2$$

$$L_q = L_s - \frac{\lambda^1}{\mu}$$
$$= L_s - \frac{\lambda(1-P_N)}{\mu}$$

$$P_N = \left[\frac{1-\rho}{1-\rho^{N+1}}\right]\rho^N$$
$$= \left[\frac{1-0.6944}{1-(0.6944)^{11}}\right](0.6944)^{10}$$
$$= \frac{(0.3056)(0.0260)}{0.9818}$$
$$= 0.00809$$

$$L_q = L_s - \frac{\lambda(1-P_N)}{\mu} = 2.06 - \frac{(0.0277)(1-0.00809)}{0.04}$$
$$= 2.06 - 0.6868$$
$$L_q = 1.373$$

Example 7.13. The arrival of cars follows Poisson distribution at a car parking place with a mean of 20 cars per hour. The capacity of parking place is 8 cars. Each car spends time at parking place follows exponential distribution with mean of 10 hours. Find

The number of cars, on an average in the parking place

Solution : Given that N = 8

$$\lambda = 20 \quad \text{cars / hour}$$
$$= \frac{20}{60} = \frac{1}{3} = 0.33 \text{ cars/ minutes}$$
$$\mu = \frac{1}{10 \times 60}$$
$$\rho = \frac{\lambda}{\mu} = \frac{10 \times 60}{3} = 200$$

Number of

L_s =

=

=

=

=

Example 7.14. I
there are 10 chair
next shop. The a
service time is
customer. Find

1. Proba

Solution Given tl
Arriv

Servi

1. P_0 =

Number of cars, on an average in the parking place

$$L_s = P_0 \sum_{n=0}^{N} n \rho^n$$

$$= \left[\frac{1-\rho}{1-\rho^{N+1}}\right] \sum_{n=0}^{N} n \rho^n$$

$$= \left[\frac{1-0.33}{1-(0.33)^{8+1}}\right] \sum_{n=0}^{8} n (0.6706)^n$$

$$= (0.6706)\ (5.2612)$$

$$= 3.5281$$

Example 7.14. In a barber shop, he can cut hair of only one person at a time and there are 10 chairs to sit. If a customer comes to shop and finds it full he goes to the next shop. The arrival rate of the customers to the shop is 10 per hour and the service time is negative exponential with an average of $\frac{1}{\mu} = 2$ minutes per customer. Find

1. Probability that the barber shop has no customer i.e., P_0

Solution Given that N = 10

Arrival rate $\lambda = 10$ customers / hour

$$= \frac{10}{60} = \frac{1}{6} = 0.166 \text{ customers / minutes}$$

Service rate $\mu = \frac{1}{2}$ minutes / customers

$$= 0.5$$

$$\rho = \frac{\lambda}{\mu} = \frac{0.166}{0.5} = 0.332$$

1. $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$

$$= \frac{1-0.332}{1-(0.332)^{10+1}}$$

$$= \frac{0.668}{1-0.000005} = \frac{0.668}{0.999} = 0.668$$

Example 7.15. Assembled computer systems are inspected in the assembly line. The arrival rate of the computer systems follows Poisson distribution and it is 20 systems per hour. The inspection rate also follows Poisson distribution with a mean of 25 systems per hour. Find

1. Average waiting number of systems in the queue for inspection in front of the inspection station as well as in the system.
2. Average waiting time per system in the queue in front of the inspection station as well as in the system.

If the waiting space is sufficient for a maximum of 5 computer systems

Solution : Given that

The arrival rate of the computer system is

$$\lambda = 20 \text{ systems per hour}$$

The inspection rate or service rate

$$\mu = 25 \text{ systems per hour}$$

Number of waiting space N = 6 computer system

$$\rho = \frac{\lambda}{\mu} = \frac{20}{25} = 0.8$$

1. Average waiting number of systems in the queue for inspection

$$L_q = L_s - \frac{\lambda^1}{\mu}$$

Where

$$\lambda^1 = \lambda(1 - P_N) \text{ and } P_N = \left[\frac{1-\rho}{1-\rho^{N+1}}\right]\rho^N$$

$$P_N = \left[\frac{1-0.8}{1-(0.8)^{5+1}}\right](0.8)^5$$

$$= \frac{(0.2)(0.3276)}{0.7378} \quad = 0.0888$$

$$\lambda^1 = \lambda(1 - P_N) = 20(1 - 0.0888) = 18.224$$

$$L_s = \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})}$$

$$= \frac{(0.8)[1-(5+1)(0.8)^5 + 5(0.8)^{5+1}}{(1-0.8)(1-0.8^{5+1})}$$

$$= \frac{(0.8)[1-1.9660+1.3107]}{(0.2)(0.7378)} \quad = 1.868$$

$$L_q = L_s - \frac{\lambda'}{\mu} = 1.868 - \frac{18.224}{25} = 1.868 - 0.7289 = 1.139$$

2. Average waiting time per computer system in the queue

$$W_q = W_s - \frac{1}{\mu}$$

Where

$$W_s = \frac{L_s}{\lambda^1}$$

$$= \frac{1.868}{\lambda(1-P_N)} = \frac{1.868}{20(1-0.088)} = \frac{1.868}{18.224} = 0.1025 \text{ hours}$$

$$W_q = W_s - \frac{1}{\mu}$$

$$= 0.1025 - \frac{1}{25}$$

$$= 0.1025 - 0.04$$

$$= 0.0625 \text{ hours}$$

Example 7.16. In big bazaar with one cashier. Assume Poisson arrivals and exponential service times. Suppose that 10 customers arrive on the average every 5 minutes and the cashier can serve 12 in 5 minutes. Find

1. The average number of customers queuing for service
2. Average time a customer spends in the system
3. Average time a customer waits before being served
4. The probability of having more than 10 customer in the system
5. The probability that a customer has to queue for more than 2 minutes.

Solution : Given that

Arrival rate is $\lambda = 10$ customers 5 minutes

$$= \frac{10}{5} = 2 \text{ customers / minutes}$$

Service rate is $\mu = 12$ customers 5 minutes

$$= \frac{12}{5} = 2.4 \text{ customers / minutes}$$

$$\rho = \frac{\lambda}{\mu} = \frac{2}{2.4} = 0.833$$

1. The average number of customers queuing for service

$$L_s = \frac{\rho}{1-\rho} = \frac{0.833}{1-0.833} = \frac{0.833}{0.167} = 4.98 = 5$$

2. Average time a customer spends in the system

$$W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{2.4(1-0.833)} = \frac{1}{(2.4)(0.167)} = 2.49$$

$$\approx 2.5 \text{ minutes}$$

3. Average time a customer waits before being served

$$W_q = \frac{\rho}{\mu(1-\rho)} = \frac{0.833}{2.4(1-0.833)}$$

$$= \frac{0.833}{(2.4)(0.167)}$$

$$= 2.07 \text{ minutes}$$

4. The probability of having more than 10 customers in the system

$$P(\geq 10) = \rho^{10}$$

$$= (0.833)^{10}$$

5. The probability that a customer has 10 queue for more than 2 minutes

$$P(W \geq 2) = (1-\rho)\rho \int_0^2 \mu e^{-\mu t(1-\rho)}\, dt$$

$$= (1-0.833)(0.833)\int_0^2 e^{-2.4(1-0.833)t}\, dt$$

$$= 0.139 \int_0^2 e^{-0.400t}\, dt$$

$$= 0.139\left[\frac{e^{0.400t}}{0.400}\right]_0^2$$

$$= 0.139\left[\frac{e^{0.900\times 2}}{0.400} - \frac{e}{0.40}\right]$$

$$= 0.139\left[\frac{0.22}{0.400} - \frac{1}{0.400}\right]$$

$$= 0.425$$

Example 7.17. Bike arrives at a drive in a shop with a mean arrival rate of 20 bikes per hour and the service rate of the bike is 15 bikes per hour. The arrival rate and the service rate follow Poisson distribution. The number of parking space for bike is only 2. Find L_s, L_q, W_q, W_s.

Solution Given that

Arrival rate $\lambda = 20$ bike / hour

Service rate $\mu = 15$ bike / hour

$$N = 2$$

$$\rho = \frac{\lambda}{\mu} = \frac{20}{15} = 1.333$$

1. $$L_s = \frac{\rho}{1-\rho^{N+1}}\left[\frac{1-(N+1)\rho^N + N\rho^{N+1}}{1-\rho}\right]$$

$$= \frac{1.333}{1-(1.333)^{2+1}}\left[\frac{1-(2+1)(1.333)^2 + 2(1.333)^{2+1}}{1-1.333}\right]$$

$$= \frac{1.333}{-1.3685}\left[\frac{1-5.3306+4.737}{-0.333}\right]$$

$$= 0.9740 \times \frac{0.4063}{0.333}$$

$$= 1.1884$$

Or $$L_s = \left[\frac{1-\rho}{1-\rho^{N+1}}\right]\sum_{n=0}^{N} n\rho^n$$

$$= \frac{1-1.333}{1-(1.333)^{2+2}}\left[\sum_{n=0}^{2} n \ (1.333\right]$$

$$= 0.2433[0.(1.333)^0 + 1.(1.333)^1 + 2.(1.333)^2]$$

$$= 0.2433[1.333 + 3.5537] \qquad = 1.188$$

2. Expected number of customers waiting in the queue

$$L_q = L_s - \frac{\lambda^1}{\mu}$$

Where λ^1 is effective arrival rate

$$\lambda^1 = \lambda(1 - P_N)$$

and $$P_N = \left[\frac{1-\rho}{1-\rho^{N+1}}\right]\rho^N$$

$$= \left[\frac{1-1.333}{1-(1.333)^{2+1}}\right](1.333)^2$$

$$= (0.2433)(1.7768)$$

$$= 0.4323$$

$$\lambda^1 = \lambda(1 - P_N)$$

$$= 20(1 - 0.4323)$$

$$= 11.352$$

$$L_q = 1.1888 - \left(\frac{11.352}{15}\right)$$

$$= 1.188 - 0.7568$$

$$= 0.4312 \text{ bike}$$ **Ans.**

3. Expected waiting time of a customer in the queue

$$W_q = W_s - \frac{1}{\mu}$$

$$= 0.1046 - \frac{1}{15}$$

$$= 0.1046 - 0.066$$

$$= 0.0379 \text{ hours}$$ **Ans.**

4. Expected waiting time of a customer in the system

$$W_s = \frac{L_s}{\lambda^1}$$

$$= \frac{1.188}{11.352} = 0.1046 \text{ hours}$$ **Ans.**

Example 7.18. The arrival rate of the lorry at a weighing station which has single weighing bridge follows Poisson distribution with mean of 50 lorries / hour. The service rate also follows Poisson distribution with mean of 52 lorries per hour. The number of parking space for lorries is 5 vehicles. Find the following

1. Average waiting number of lorries in the queue in front of the weighing bridge.
2. Average waiting time per lorry in front of the weighing bride
3. Average waiting number of lorries in the system
4. Average waiting time per lorry in the weighing station

Solution : Given that

The arrival rate $\lambda = 50$ lorries / hours

The service rate $\mu = 52$ lorries / hours

The number of parking space for lorries N = 5

1. Average waiting number of lorries in the queue

$$L_q = L_s - \frac{\lambda^1}{\mu}$$

Where

$$\lambda^1 = \lambda(1 - P_N)$$

$$\rho = \frac{\lambda}{\mu} = \frac{50}{52} = 0.9615$$

$$P_N = \left[\frac{1-\rho}{1-\rho^{N=1}}\right]\rho^N$$

$$= \left[\frac{1-0.9615}{1-(0.9615)^{5+1}}\right](0.9615)^5$$

$$= \frac{0.0385}{0.2098} \times 0.8217$$

$$= 0.1508$$

$$\lambda^1 = \lambda(1 - P_N)$$

$$= 50(1 - 0.1508)$$

$$= 42.46$$

2. $$W_q = W_s - \frac{1}{\mu}$$

$$= 0.0563 - \frac{1}{52}$$

$$= 0.0563 - 0.0192$$

$$W_q = 0.03706$$

3. Average waiting number of lorries in the system

$$L_s = \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})}$$

$$= \frac{(0.9615)[1-(5+1)(0.9615)^5 + 5(0.9615)^{5+1}]}{(1-0.9615)(1-0.9615^{5+1})}$$

$$= \frac{(0.9615)[1-6\times 0.82176+5\times 0.79012]}{(0.0385)(0.2098)}$$

$$= \frac{(0.9615)[1-4.93050+3.9506]}{8.0773\times 10^{-3}}$$

$$= \frac{0.01932}{8.0773\times 10^{-3}}$$

$$= 2.391$$

$$L_q = L_s - \frac{\lambda^1}{\mu}$$

$$= 2.391 - 0.8165$$

$$= 1.5745$$

4. $$W_s = \frac{L_s}{\lambda^1} = \frac{2.391}{42.46} = 0.0563$$

Example 7.19. The arrival of cars follows Poisson distribution at a car parking place with a mean of 20 cars per hour. The capacity of parking place is 8 cars. Each car spends time at parking place follows exponential distribution with mean of 10 hours. Find the number of cars, on an average in the parking place

Solution : Given that N = 8

$$\lambda = 20 \text{ cars / hour}$$

$$= \frac{20}{60} = \frac{1}{3} = 0.33 \text{ cars / minutes}$$

$$\mu = \frac{1}{10\times 60}$$

$$\rho = \frac{\lambda}{\mu} = \frac{10\times 60}{3} = 200$$

Number of cars, on an average in the parking place

$$L_s = P_0 \sum_{n=0}^{N} n\rho^n$$

$$= \left[\frac{1-\rho}{1-\rho^{N+1}}\right] \sum_{n=0}^{N} n\rho^n$$

$$= \left[\frac{1-200}{1-200^{8+1}}\right]\sum_{n=0}^{8} n(200)^n$$

$$= 3.88\times 10^{-19}[0\times(200)^0 + 200 + 2(200)^2 + 3(200)^3 + 4(200)^4$$

$$+ 5(200)^5 + 6(200)^6 + 7(200)^7 + 8(200)^8$$

$$= 3.88\times 10^{-19}\,[2.056\times 10^{+19}]$$

$$= 7.99 \quad \approx 8$$

$$L_s = \frac{\rho}{1-\rho^{N+1}}\left[\frac{1-(N+1)\rho^N + N\rho^{N+1}}{1-\rho}\right]$$

$$= \frac{200}{1-200^{8+1}}\left[\frac{1-(8+1)200^8 + 8(200)^{8+1}}{1-200}\right]$$

$$= \frac{200}{-5.12\times 10^{20}}\left[\frac{1-2.304\times 10^{19} + 4.096\times 10^{21}}{-199}\right]$$

$$= 3.90625\times 10^{-19}\left[\frac{4.07296\times 10^{21}}{199}\right]$$

$$= [3.90625\times 10^{-19}][2.0467\times 10^{19}]$$

$$= 7.99$$

$$\approx 8$$

:ar parking
cars. Each
nean of 10

Example 7.20. Patients arrive at a clinic according to a Poisson distribution and the arrival rate is 20 patients per hour. Examination time per patient is exponential with mean rate of 15 per hour. In the waiting room accommodation facility is for only 10 patients. Find

1. The effective arrival rate at the clinic
2. The probability that an arriving patient will not wait
3. The expected waiting time until a patient is discharged from the clinic

Solution : Given that

The arrival rate at a clinic

$\lambda = 20$ patients per hour

The examination time or service rate

$\mu = 15$ per hour

Accommodation facility in the waiting room

$N = 10$ patients

1. The effective arrival rate at the clinic

$$\lambda^1 = \lambda(1-P_N) \text{ or } \lambda^1 = \mu(L_s - L_q)$$

We know that

$$P_N = \left[\frac{1-\rho}{1-\rho^{N+1}}\right]\rho^N \text{ where } \rho = \frac{\lambda}{\mu} = \frac{20}{15} = 1.333$$

$$= \left[\frac{1-1.333}{1-1.333^{10+1}}\right](1.333)^{10} = \frac{(0.333)}{(22.61)}(17.75) = 0.2614$$

$$\lambda^1 = \lambda(1-P_N)$$

$$= 20(1-0.2614)$$

$$= 14.77$$

2. The probability that an arriving patient will not wait

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-1.333}{1-(1.333)^{10+1}} = \frac{0.333}{22.61}$$

$$P_0 = 0.0147$$

3. The expected waiting time until a patient is discharged from the clinic

$$w_s = \frac{L_s}{\lambda^1}$$

$$L_s = \frac{\rho}{1-\rho^{N+1}}\left[\frac{1-(N+1)\rho^N + N\rho^{N+1}}{1-\rho}\right]$$

$$= \frac{1.333[1-(10+1)(1.333)^{10} + 10(1.333)^{10+1}]}{(1-1.333)(1-1.333^{10+1})}$$

$$= \frac{1.333[1-194.84+236.11]}{(0.333)(22.61)}$$

$$= \frac{(1.333)(42.27)}{(0.333)(22.61)}$$

$$= 7.48$$

$$W_s = \frac{L_s}{\lambda^1} = \frac{7.48}{14.77} = 0.506 \quad hours$$

Example 7.21. Pa
distribute and arr
accommodate mo
mean rate of 20 pe

1) The ef
2) Find th
3) The ex

Solution Given th
Arriva

Meetin

1) The ef
λ^1
P_N

λ^1

2) The pr

Example 7.21. Parents arrive at a college to meet principal according to Poisson distribute and arrival rate is 30 parents per hour. The waiting room does not accommodate more than 14 Parents. Meeting time per parent is exponential with mean rate of 20 per hour. Find

1) The effective arrival rate at the college
2) Find the probability that an arriving parent will not wait.
3) The expected waiting time until a parent will leave the college.

Solution Given that:

Arrival rate of the parent is

$$\lambda = 30 \text{ parents per hour}$$

Meeting time or service rate of the parents is

$$\mu = 20 \text{ parents hour}$$

$$\rho = \frac{\lambda}{\mu} = \frac{30}{20} = 1.5$$

1) The effective arrival rate at the college

$$\lambda^1 = \lambda(1 - P_N)$$

$$P_N = [\frac{1-\rho}{1-\rho^{N+1}}]\rho^N$$

$$= [\frac{1-1.5}{1-1.5^{14+1}}](1.5)^{14}$$

$$= [\frac{0.5}{436.89}](291.92)$$

$$= 0.3340$$

$$\lambda^1 = \lambda(1 - P_N)$$

$$= 30(1 - 0.3340)$$

$$= 19.977$$

$$\approx 19.98$$

2) The probability that an arriving parent will not wait

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-1.5}{1-1.5^{14+1}}$$

$$= \frac{0.5}{436.89}$$
$$= 1.14 \times 10^{-3}$$
$$= 0.00114$$

3) The expected waiting time until a parent leave the college

$$w_s = \frac{L_S}{\lambda^1}$$

$$L_S = \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})}$$

$$= \frac{1.5[1-(14+1)(1.5)^{14} + 14(1.5)^{14+1}]}{(1-1.5)(1-1.5^{14+1})}$$

$$= \frac{1.5[1-4378.93+6130.51]}{(0.5)(436.89)}$$

$$= 12.03$$

$$w_s = \frac{L_S}{\lambda^1} = \frac{12.03}{19.98} = 0.6023 \textit{ hours}$$

Example 7.22. Cars arrive at a parking place in big bazaar according to a Poisson distribution with mean 5 cars per hour. The time for keeping each car at the parking place, varies but is found to follow an exponential distribution with mean 10 minutes per car. Parking place has space of only 5 cars. Find

1. The effective arrival rate
2. The expected number of parking spaces occupied

Solution : Given that

The arrival rate at a parking place $\lambda = 5\, cars / hour$

The time for keeping each car at the parking place or service time

$$\mu = 10 \min utes / car$$
$$= 6\, cars / hour$$

$$\rho = \frac{\lambda}{\mu} = \frac{5}{6} = 0.8333$$

The effective arrival rate

$$\lambda^1 = \lambda(1 - P_N)$$

$P_N =$

$\lambda^1 =$

$=$

$=$

$=$

The expected

Example 7.23. A
service distributio
is sufficient only 1
The arrival rate of
is with mean of 12

1. The steady s
2. The probabil
3. The average

Solution : Given t

The ar

The se

The ma

The pr

$$P_N = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.8333}{1-(0.8333)^{5+1}} = \frac{0.1666}{0.6651} = 0.2504$$

$$\lambda^1 = \lambda(1-P_N)$$

$$= 5(1-0.2504)$$

$$= 5(0.749)$$

$$= 3.747$$

The expected number of parking spaces occupied

$$L_q = \frac{\rho^2[1-N(\rho)^{N-1}+(N-1)\rho^N]}{(1-\rho)[1-\rho^{N=1}]}$$

$$= \frac{(0.8333)^2[1-5(0.8333)^{5-1}+(5-1)(0.8333)^5]}{(1-0.8333)[1-(0.8333)^5]}$$

$$= \frac{0.6943[1-2.4108+106071]}{0.1667(0.6651)}$$

$$= 1.229$$

Ans.

Example 7.23. At a railway station, assuming Poisson arrivals and exponential service distribution, only one train is handled at a time, the space in the railway yard is sufficient only for 4 trains to wait while other is given signal to leave the station. The arrival rate of trains at the station with mean of 6 per hour and the service rate is with mean of 12 per hour. Find

1. The steady state probabilities for the various number of trains in the system
2. The probability that there is no train in the system (both waiting and in service)
3. The average waiting time of a new train coming into the yard.

Solution : Given that

The arrival rate of trains at the station

$\lambda = 6$ per hour

The service rate is $\mu = 12$ per hour

$$\rho = \frac{\lambda}{\mu} = \frac{6}{12} = 0.5$$

The maximum number of trains in the system N = 5

$$P_n = P_0\rho^n$$

The probability that there is no train in the system

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.5}{1-(0.5)^6} = \frac{0.5}{0.9} = 0.508$$

$$P_n = P_0 \rho^n$$

$P_1 = P_0\rho^1$	$P_2 = P_0\rho^2$	$P_3 = P_0\rho^3$	$P_4 = P_0\rho^4$	$P_5 = P_0\rho^5$
$P_1 = 0.508(0.5)^1$	$P_2 = 0.508(0.5)^2$	$P_3 = 0.508(0.5)^3$	$P_4 = 0.508(0.5)^4$	$P_5 = 0.508(0.5)^5$
$P_1 = 0.254$	$P_2 = 0.127$	$P_3 = 0.0635$	$P_4 = 0.0317$	$P_5 = 0.0158$

$$L_s = P_0 \sum_{n=0}^{N} n\rho^n$$

$$= P_0[\rho^1 + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5]$$

$$= P_0\rho^1 + 2P_0\rho^2 + 3P_0\rho^3 + 4P_0\rho^4 + 5P_0\rho^5$$

$$= 0.254 + 2\times 0.127 + 3(0.0635) + 4(0.0317) + 5(0.0158)$$

$$= 0.254 + 0.254 + 0.1905 + 0.1268 + 0.079$$

$$L_s = 0.9043$$

Or

$$L_s = \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})}$$

$$= \frac{(0.5)[1-(5+1)(0.5)^5 + 5(0.5)^{5+1}]}{(1-0.5)(1-0.5^{5+1})}$$

$$= \frac{(0.5)[1-0.1875+0.07812]}{(0.5)(0.9843)}$$

$$L_s = 0.9048$$

Thus the average number of trains in the system is 0.9048

Each train takes on an average $\frac{1}{12} = 0.08$ hours for getting service.

3. The average waiting time of a new train coming into the yard.

$$W_q = (0-0.9048)(0.080) = 0.0723 hours$$

or

$$\lambda^1 = \lambda(1-P_N) \text{ and } P_N = \left[\frac{1-\rho}{1-\rho^{N+1}}\right]\rho^N$$

$$= \left[\frac{1-0.5}{1-0.5^{5+1}}\right]0.5^5$$

$$= \left[\frac{1-0.5}{1-0.015625}\right]0.03125$$

$$= \left[\frac{0.5}{0.984375}\right]0.03125$$

$$= 0.0158730$$

$$\lambda^1 = \lambda(1-P_N)$$

$$= 6(1\text{-}0.0158730)$$

$$= 5.90476$$

Where

$$W_s = \frac{L_s}{\lambda^{'}} = \frac{0.9048}{5.90476} = 0.153232$$

$$W_q = W_s - \frac{1}{\mu}$$

$$= 0.153232\text{- }(1/12)$$

$$= 0.069902$$

$$\approx 0.07 \text{ hours}$$

Example 7.24. In the following examples of queuing system identify the customers and the servers.

1. The bicycle repair shop
2. The materials – handling equipment in a factory area
3. The toll gate
4. Ration shop
5. Airport runways
6. Telephone booth
7. Maintenance shop
8. Traffic signal
9. A fire station
10. A plumbing shop

Solution

Customers		**Servers**
1.	Bicycles	Repair man
2.	Materials	Material handling equipment
3.	Vehicles	Toll collectors
4.	Ration card holders	Shopkeeper
5.	Planes	Airport Runways
6.	Customers	Telephone Booth
7.	Breakdown machines	Mechanics
8.	Vehicles	Signal point
9.	Fire tender	Filling station
10.	Customers	Plumbing shop

EXERCISE

Model I :(M/M/1) : (GD/∞/∞)

1) The arrival rate of patients at a clinic follows Poisson distribution with a mean of 30 per hour. The service rate of a doctor also follows Poisson distribution with a mean of 50 per hour. Find.

a) The probability of having 0 patients in the system. i.e., P_0

b) The probability of having 5 patients in the system i.e., P_5

c) Average number of waiting patients in the system i.e., L_s

d) Average number of waiting patients in the queue i.e., L_q

e) Average waiting time of patients in the system i.e., W_s

f) Average waiting time of patients in the queue i.e., W_q

[**Hint**: Arrival rate, $\lambda = 30$ per hour, Service rate, $\mu = 50$ per hour

The utilization factor $\rho = \lambda/\mu = 30/50 = 0.6$

a) The probability of having 0 patient in the system $P_0=1-\rho=1-0.6=0.4$ **Ans.**

b) The probability of having 5 patients in the system $P_5 = (1-\rho)\,\rho^5 = (1\text{-}0.6)\,(0.6)^5=0.031104$ **Ans.**

c) Average number of waiting patients in the system $L_s= \rho/(1-\rho)=0.6/(1-0.6)=1.5$patients **Ans.**

d) Average number of waiting patients in the queue $L_q= \rho^2/(1-\rho)= (0.6)^2/(1-0.6)=0.9$ patients **Ans.**

e) Average waiting time of patients in the system $W_s=1/\mu\,(1-\rho)=0.05$ **Ans.**

f) Average waiting time of patients in the queue $W_q= \rho/\mu\,(1-\rho)= 0.03$ **Ans.**]

2) On an average of 10 programmes are run every 5 minutes on a computer, while the computer can run 12 programmes in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service rate (output from computer) find.

1. Averag
2. Averag
3. Averag
4. Averag

[**Hint:** Arrival rate $\lambda =$

1.Average numl

2.Average numb

3. Average time

=

4. Average time

3) The service ti
rate follows Poisso
The arrival rate of
service rate are 40 c

1. Utilizat
2. Averag
3. Averag
4. Averag
5. Averag

[**Hint:** The arrival rate
per hour And the se

1. Utilizatio
2. Average
$L_q= \rho^2$
3. Average
$L_s= \rho/(1$
4. Average
$W_q=$
5. Average

1. Average number of programmes in queue
2. Average number of programmes in the system
3. Average time a programme takes before getting output.
4. Average time a programme takes in the system.

[**Hint:** Arrival rate $\lambda = \frac{10}{5} = 2$ programmes / minute, Service rate μ=12/5=2.4 programmes/ minute

1.Average number of programme in queue

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(2)^2}{2.4(2.4-2)} = \frac{4}{(2.4)(0.4)} = 4.166$$

2.Average number of programmes in the system $= \frac{\lambda}{\mu-\lambda} = \frac{2}{2.4-2} = \frac{2}{0.4} = 5$

3. Average time a programme takes before getting output

$$= \frac{\lambda}{\mu(\mu-\lambda)} = \frac{2}{2.4(2.4-2)} = \frac{2}{(2.4)(0.4)} = 2.0833 \text{ minutes}$$

4. Average time a programme takes in the system $= \frac{1}{\mu-\lambda} = \frac{1}{2.4-2} = \frac{1}{0.4} = 2.5$ minutes **Ans.**]

3) The service time follows exponential (negative) distribution and the service rate follows Poisson distribution at a single window to deposit money in a bank. The arrival rate of customers follows Poisson distribution. The arrival rate and the service rate are 40 customers per hour and 50 customers per hour respectively. Find

1. Utilization of the depositing clerk
2. Average number of waiting customers in the queue
3. Average number of waiting customers in the system
4. Average waiting time per customers in the queue
5. Average waiting time per customers in the system

[**Hint**: The arrival rate of customers at a single window in a bank to deposit money λ=40 customers per hour And the service rate μ=50 customers per hour

1. Utilization of the depositing clerk $\rho = \lambda / \mu = 40/50 = 0.8$ **Ans.**
2. Average number of waiting customers in the queue

   $L_q = \rho^2/(1-\rho) = (0.8)^2/(1-0.8) = 3.2$ customers **Ans.**
3. Average number of waiting customer in the system

   $L_s = \rho/(1-\rho) = 0.8/(1-0.8) = 4$ customers **Ans.**
4. Average waiting time per customer in the queue

   $W_q = \rho/(\mu-\lambda) = 0.8/50-40 = 0.08$ hour **Ans.**
5. Average waiting time per customer in the system

   $W_s = 1/\mu-\lambda = 1/50-40 = 0.1$hour **Ans.**]

4) Customers arrive at the rate of 30 customers per hour in a barber shop. That shop has a single main he can cut hair of only one person at a time. The average number of customers that can be serviced by the barber is 34 per hour. Calculate

1. The probability that the barber is idle
2. The average number of customers in the queuing system
3. The average number of customers in the queue
4. The average time a customer spends in the system
5. The average time a customer spends in the queue waiting for service.

[Hint: $\lambda = 30$ customers / hour, $\mu = 34$ customers / hour

1. The probability that the barber is idle The utilization factor $\rho = \frac{\lambda}{\mu} = \frac{30}{34} = 0.882$ **Ans.**

$$P_0 = 1 - \rho = 1 - 0.882 = 0.118 \text{ Ans.}$$

2. The average number of customers in the queuing system

$$L_s = \frac{\rho}{1-\rho} = \frac{0.882}{1-0.882} = \frac{0.882}{0.118} = 7.474 \text{ Ans.}$$

3. The average number of customer in the queue

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.882)^2}{1-0.882} = \frac{0.7779}{0.118} = 6.592 \text{ Ans.}$$

4. The average time the customer spends in the system

$$W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{34(1-0.882)} = \frac{1}{34(0.118)} = 0.249 \text{ Hours} = 14.97 \text{ minutes Ans.}$$

5. The average time a customer spends in the queue waiting for service

$$W_q = \frac{\rho}{\mu(1-\rho)} = \frac{0.882}{34(1-0.882)} = \frac{0.882}{34 \times 0.118} = 0.2198 \text{ hour's} = 13.18 \text{minutes Ans.]}$$

Model II :(M/M/1): (GD/N/∞)

5) In a railway marshalling yard, goods trains arrive at the rate of 35 trains per day, inter arrival time follows an exponential distribution. Service time is also follows an exponential distribution with mean of 15 minutes. Find

1. Probability that the yard is empty
2. Average queue length, given that, the capacity of the yard is 10 trains

[Hint: λ=35 trains / day=35/60×24=7/288 = 0.02430 trains / minutes μ =1/15 Trains / minutes

$\rho = \lambda/\mu = (15\times7)/288 = 0.3645$

1. Probability that the yard is empty

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} \qquad \text{Where N = 10, } P_0 = 0.6355 \textbf{ Ans.}$$

2. Average queue length

$$L_s = \frac{\rho}{1-\rho^{N+1}}\left[\frac{1-(N+1)\rho^N + N\rho^{N+1}}{1-\rho}\right], = 0.57347,$$

$$P_N = \left[\frac{1-\rho}{1-\rho^{N+1}}\right]\rho^N = 0.0000262$$

$$L_q = L_s - \frac{\lambda(1-P_N)}{\mu} = 0.20898 \textbf{ Ans.]}$$

6) In a car workshop, mechanic can repair only one car at a time and there are 10 cars in the workshop. Only 10 cars can be parked in the workshop. If a customer comes to workshop and finds it full, he goes to the next workshop. The arrival rate of the customers to the shop is 10 per hour and the service time is negative exponential with an average of $\frac{1}{\mu} = 2$ minutes per customer. Find

1. Probability that the workshop has no customer i.e. P_0

[**Hint**: N = 10, Arrival rate $\lambda = 10$ customers / hour $= \frac{10}{60} = \frac{1}{6} = 0.166$ customers / minutes

Service rate $\mu = \frac{1}{2}$ minutes / customers $= 0.5$, $\rho = \frac{\lambda}{\mu} = \frac{0.166}{0.5} = 0.332$

1. $$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.332}{1-(0.332)^{10+1}} = \frac{0.668}{1-0.000005} = \frac{0.668}{0.999} = 0.668 \textbf{ Ans.]}$$

OBJECTIVE TYPE QUESTIONS

1. Queue can form only when

   (a) Arrivals equals number of services

   (b) Arrivals exceed service capacity

   (c) Service facility is capable to serve all the arrivals

   (d) There are two service facilities

Ans. (b)

2. Customer behavior in which he moves from one queue to another with more than one server is known as

   (a) Reneging (b) Jockeying (c) Balking (d) None if them

Ans. (b)

3. Multiple servers always will be in series, Statement is

   (a) True (b) False

   (c) Both (a) & (b) correct (d) none if the above

Ans. (b)

4. Multiple servers may be

   (a) In Series (b) In combination of Parallel and Series

   (c) In Parallel (d) All of the above

Ans. (d)

5. For a Poisson exponential single server and infinite population queuing model

   (a) The arrivals occur in a Poisson fashion

   (b) The system has a single service facility

   (c) Both (a) & (b) are correct (d) None of the above

Ans. (c)

6. For a Poisson exponential single server and infinite population queuing model which of the following is correct

   (a) $E(n) = E(m) - \lambda/\mu$ (b) $E(v) = E(w) - 1/\mu$

   (c) $E(m) = \lambda E(w)$ (d) $E(n) = E(v)$

Ans. (c)

7. Which of the following is correct?

   (a) The distribution of waiting time is not related to queue discipline used in selecting the waiting customers for service.

   (b) The probability of a n customers arriving during a time interval t, according to Poisson law is given by $P(n)\ e^{-\lambda t}(\lambda t)^n/n!$

   (c) When the waiting customer becomes impatient and decides to leave the queue, the customer is said to have balk.

   (d) In the generalized queuing model, an arrival can be considered as a death, where as a departure can be considered as births.

Ans. (b)

8. Which of

   (a) The c
   come

   (b) The e
   of qu

   (c) Both

Ans. (a)

9. The formu

   (a) Mean S

   (c) Mean S

   (d) Mean A

Ans. (d)

10. The patter

   (a) Service

   (c) Service

Ans. (c)

11. If the custo

   (a) Balking

Ans. (c)

12. The Traffic

   (a) < 1

Ans. (a)

13. When the b
   is called

   (a) Transie

Ans. (b)

14. If the opera
   called

   (a) Transie

Ans. (a)

15. The Probab

   (a) 1-Δt

Ans. (b)

8. Which of the following is not correct?
   (a) The only way the customers are serviced in queuing situations is the first come – first service basis.
   (b) The expected length of the system should be equal to the expected length of queue plus one.
   (c) Both (a) & (b) not correct (d) None of the above

Ans. (a)

9. The formula of the traffic intensity is
   (a) Mean Service rate (b) Mean Arrival rate
   (c) Mean Service rate/ Mean Arrival rate
   (d) Mean Arrival rate/ Mean Service rate

Ans. (d)

10. The pattern according to which the customers are served is called
    (a) Service mechanism (b) Service rate
    (c) Service discipline (d) Arrival rate

Ans. (c)

11. If the customer leaves the counter due to some reason, it is called
    (a) Balking (b) Jockeying (c) Reneging (d) Priority

Ans. (c)

12. The Traffic intensity ρ is
    (a) < 1 (b) > 1 (c) $= 1$ (d) $e^{-\lambda t}$

Ans. (a)

13. When the behavior of the system becomes independent of time then the state is called
    (a) Transient State (b) Steady State (c) Explosive State (d) None of them

Ans. (b)

14. If the operating characteristics (behavior) dependent on time then the state is called
    (a) Transient State (b) Steady State (c) Compressible State (d)None of them

Ans. (a)

15. The Probability of exactly one arrival in Δt time is given by
    (a) $1-\Delta t$ (b) $\lambda\Delta t$ (c) $P_n t\,(1-\Delta t)$ (d) None of them

Ans. (b)

16. The Probability of no arrival in Δt time is given by

    (a) $1-\Delta t$ (b) $\lambda\Delta t$ (c) $\lambda\Delta t-1$ (d) None of them

Ans. (a)

17. The model in which only arrivals are considered and no departures take place are called

    (a) Pure Death model (b) Pure Birth model

    (c) Both (a) & (b) (d) None of them

Ans. (a)

18. The model in which only departures are considered and no arrival take place are called

    (a) Pure Death model (b) Pure Birth model

    (c) Pure Death model/ Pure Birth model (d) None of them

Ans. (a)

19. The Probability that one departure takes place in the small interval of time is given by

    (a) $\mu\Delta t$ (b) $\mu\Delta t^2$ (c) $1-\mu\Delta t$ (d) None of them

Ans. (a)

20. In the model of (M/M/1):(∞/FIFO),the expected number of customers/units in the system is given by

    (a) $L_s = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$ (b) $L_q = \frac{\rho^2}{1-\rho}$ (c) $L_s = \frac{1-\rho}{\rho}$ (d) None of them

Ans. (a)

21. In the reference of above question expected number of customers/units in the queue is given by

    (a) $L_q = \frac{\rho^2}{1-\rho}$ (b) $L_q = \frac{1-\rho}{\rho^2}$ (c) $L_s = \frac{\rho}{1-\rho}$ (d) None of them

Ans. (a)

22. In the reference of above question the expected waiting time per customer or unit in the system and expected waiting time per customer or unit in the queue respectively is given by

    (a) $W_s = \frac{L_s}{\lambda}$, $W_q = W_s - \frac{1}{\mu}$ (b) $W_s = \frac{\lambda}{L_s}$, $W_q = W_s - \mu$

    (c) $W_s = L_s - \lambda$, $W_q = W_s - 1/\mu$ (d) None of them

Ans. (a)

UNIT-8

STOCHASTIC PROCESSES

"Attitudes are more important than facts."

8.1 INTRODUCTION TO STOCHASTIC PROCESSES

A stochastic process is a sequence of random variables $\{x_t\}$ where $t \in T$ is time. The range space for x_t may be discrete or continuous. (In other words to analyse any mathematical model, we require not only several random variables but also several different families of random variables which are functions of time. For example, In the queuing system; waiting in a queue w (t) is a random function of time. Such random functions of time are called stochastic processes. Some types of stochastic processes are given below

8.1.1 Discrete State and Continuous state process

Discrete state and continuous state processes depend on the values the state can take. values of state may be finite (countable) or infinite. If values of state are finite or countable the process is called a discrete – state process. For example the number of customers in a queuing system can take only discrete values; it is called discrete state process. The waiting time W (t) may be infinite, is called continuous state process.

8.1.2 Markov Process

This chapter introduces a special type of stochastic process called a Markov process. If the futures state of a process depends only on the present and is

independent of the past, then the process is called a Markov process. In Markov process, next state can be analyzed based on the present state. A discrete state Markov process is called a Markov chain. For example a sequence of repeated trial of an experiment in which the out come at any step in the sequence depends at most on the out come of the preceding step and not on any other previous out come. Such a sequence is called a Markov chain or Markov process. There are two types of Markov chains

1) Discrete – Time Markov chains

2) Continuous – Time Markov chains

Examples →

1) A man either eats idly or vada each day. He never eats Idly for continuous 2days but if he eats vada then the next day he is just as likely to eat it again as he is to eat Idly.
2) A man goes to office by either Bus or Taxi. If he goes one day by Bus he is 70% sure to go by Taxi next day, but if he goes by Taxi one day he is 60% sure to go by Taxi next day.

8.2 PROBABILITY VECTOR

A vector $e = [e_1, e_2 ...., e_n]$ is called a probability vector if its all elements are positive (nonnegative) and their sum is one that is

1) Each $e_i \geq 0$ 2) $e_1 + e_2 + ... + e_n = 1$

8.3 STOCHASTIC MATRIX

A square matrix P = $[P_{ij}]$ is called a stochastic matrix if each row of P is a probability vector.

8.3.1. Theorem

If two matrices A and B are stochastic matrices, then the product A.B is also a stochastic matrix. All powers of A and all powers of B are stochastic matrices.

8.4. REGULAR STOCHASTIC MATRIX

A stochastic matrix P is said to be regular if all the elements of some power P^k of P are positive.

8.5. TRANSITION MATRIX

A Markov process or Markov chain consists of a sequence of repeated trials of an experiment with the property that

1) In the experiment of repeated trials each out come belongs to a finite set $\{b_1, b_2 .... b_n\}$ called the state space of the system.

2) In the experiment of repeated trials, the outcome of any future trial depends only on the outcome of the present (preceding) trial and not on any other previous or past trial.

For the given probability p_{ij}, if b_j occurs immediately after b_i, the following n – square matrix.

$$P = \begin{array}{c} \\ b_1 \\ b_2 \\ . \\ . \\ . \\ . \\ b_n \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & \cdots\cdots & b_n \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ p_{n1} & p_{n2} & \ldots & p_{nn} \end{bmatrix} \end{array}$$

This matrix P is called the transition matrix of Markov process.

Thus transition matrix is a rectangular array which consist the transition probabilities for a given Markov process. In transition matrix row denotes the current state of the system and the columns denote the alternative states to which the system can move.

In the transition matrix of the Markov chain if $p_{ij} = 0$, when no transition occurs from state i to j (Transition is impossible) and $p_{ij} = 1$, when the system is moving from i to j.

8.5.1. Theorem The transition matrix M of a Markov process is a stochastic matrix.

8.6 MARKOV PROCESS CLASSIFICATION OF STATES

Suppose a transition matrix P, with Markov process is

$$P = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2n} \\ . & . & & \\ . & . & & \\ . & . & & \\ . & . & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

The probability vector is

$$e = [e_1, e_2 \ldots\ldots e_n]$$

Then i the distribution of the Markov process has following probability vector

$$e_i = [e_{i1}, e_{i2} \ldots\ldots e_{in}]$$

If the initial state distribution at time t = 0 is e_0.

Then the subsequent state distribution can be abstained by multiplying the previous state distribution by the transition matrix P

$e_0\, p = e_1$

$e_1\, p = e_2$

$e_2\, p = e_3$

$e_3\, p = e_4$ ------

$e_1 = e_0 p$

$e_2 = e_1 p = (e_0 p) p = e_0 p^2$

$e_3 = e_2 p = (e_1 p) p = (e_0 pp)\ p = e_0\ p^3$

8.6.1 Theorem

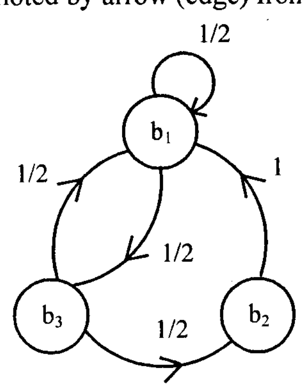
If an initial state distribution e_0 is given

Then for i = 1, 2 ---------

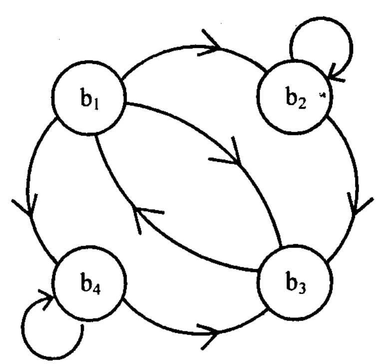
$e_i = e_{i-1}\, p = e_0 p^i$

8.7 TRANSITION DIAGRAM

Two transition diagrams are shown in the figure, it shows the transition probabilities. The states are vertices in the diagram and probability P_{ij} is denoted by arrow (edge) from state b_i to the state b_j.



(Diagram a)



(Diagram b)

Transition mat

Transition mat

Example 8.1 :

Solution

Given nor

For any v

(1) All en

(2) Additi

For the g
entries is

2 + 5 + 0

⇒ Non ze

Example 8.2 :
probability vect

Solution

For given
V, which
by multipl

$V = \frac{1}{8} e =$

Transition matrix of the diagram a,

$$
P = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array}
\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\
\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}
$$

Transition matrix of the diagram b,

$$
P = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array}
\begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\
\begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \end{array}
$$

SOLVED EXAMPLES

Example 8.1 : Show that the non zero vector e = [2,5,0,1] is the probability vector.

Solution

Given non zero vector e = [2, 5, 0, 1] is not a probability vector

For any vector to be a probability vector it should follow two conditions

(1) All entries should be non negative

(2) Addition of al entries should be one.

For the given non zero vector all entries are positive but addition of all entries is

$2 + 5 + 0 + 1 = 8 \neq 1$

$\Rightarrow$ Non zero vector e = [2, 5, 0, 1] is not a probability vector.

Example 8.2 : In the previous example for the non zero vector e = [2, 5, 0, 1] find probability vector.

Solution

For given non zero vector e = [2, 5, 0, 1] there is a unique probability vector V, which is a scalar multiple of e. This probability vector V, can be obtained by multiplying e by the reciprocal of the addition of its entries. That is

$$V = \frac{1}{8} e = \left[\frac{2}{8}, \frac{5}{8}, \frac{0}{8}, \frac{1}{8}\right]$$

Example 8.3 Check the following stochastic matrix is regular or not

$$A = \begin{bmatrix} 1 & 0 \\ 7/8 & 1/8 \end{bmatrix}$$

Solution

We know that for any stochastic matrix to be regular, all the entries or elements of some power, p^k of p should be positive for the given matrix

$$A^2 = \begin{bmatrix} 1 & 0 \\ 7/8 & 1/8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 7/8 & 1/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+0\times7/8 & 1\times0+0\times1/8 \\ 7/8\times1+1/8\times7/8 & 7/8\times0+1/8\times1/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 63/64 & 1/64 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} 1 & 0 \\ 63/64 & 1/64 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 63/64 & 1/64 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+0\times63/64 & 1\times0+0\times1/64 \\ \frac{63}{64}\times1+\frac{1}{64}\times\frac{63}{64} & \frac{63}{64}\times0+\frac{1}{64}\times\frac{1}{64} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{4095}{4096} & \frac{1}{4096} \end{bmatrix}$$

$\Rightarrow$ A is not a regular matrix, since every power A^k of A will have 1 and 0 in the first row.

Example 8.4: Suppose $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Square stochastic matrix, and e = [1/2 1/2] is a probability vector than show that eA is also a probability vector

Solution

First we have to find eA matrix by multiplying e matrix with A,

$$eA = [1/2 1/2] \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \left[\frac{1}{2}\times 0+1/2\times 1/2 \quad 1/2\times 1+1/2\times 1/2\right]$$

$$= [1/4 \quad 3/4]$$

Since the sum of the elements of (e. A) Matrix is $\frac{1}{4}+\frac{3}{4}=1$

Thus eA is a probability vector.

Example 8.5 : Determine which of the following stochastic matrices are regular

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$

Solution

(a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1+0\times 1 & 1\times 0+0\times 0 \\ 1\times 1+0\times 1 & 1\times 0+0\times 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$A^3 = A^2 \times A$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus every power of A is the matrix A. Thus every power of A has zero entries, and so A is not regular.

(b) $A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

$$=\begin{bmatrix} 1/2\times1/2+1/2\times0+0\times1/2 & 1/2\times1/2+1/2\times1+0\times1/4 & 1/2\times0+1/2\times0+0\times1/4 \\ 0\times1/2+1\times0+0\times1/2 & 0\times1/2+1\times1+0\times1/4 & 0\times0+1\times0+0\times1/4 \\ 1/2\times1/2+1/4\times0+1/4\times1/2 & 1/2\times1/2+1/4\times1+1/4\times1/4 & 1/2\times0+1/4\times0+1/4\times1/4 \end{bmatrix}$$

$$=\begin{bmatrix} 1/4 & 3/4 & 0 \\ 0 & 1 & 0 \\ 3/8 & 9/46 & 1/16 \end{bmatrix}$$

Thus A is not regular.

Matrix A, has one in the diagonal element

$\Rightarrow$ A is not regular.

Example 8.6: Given $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ & $u = [1 \quad 3 \quad -2]$ find u. A

Solution

$$u\,A = [1 \quad 3 \quad -2] \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= [1\times 1+3\times 2+(-2)\times 1 \quad 1\times 3+3\times 1+(-2)\times 1 \quad 1\times (-1)+3\times 2+(-2)\times 1]$$

$$= [1+6-2 \quad 3+3-2 \quad -1+6-2]$$

$$= [5 \quad 4 \quad 3]$$

Example 8.7 : Find the unique fixed probability vector S of each matrix

(a) $A = \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix}$

Solution

Suppose probability vectors

$S = [x \quad 1-x]$ s. t. S. A = S

$$[x \quad 1-x] \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (x \quad 1-x)$$

$$[0.3x+(1-x)0-5 \quad 0.7x+(1-x)0.5] = [x \quad 1-x]$$

$$[0.3x+0-5-0.5x \quad 0.7x+0.5-0.5x] = [x \quad 1-x]$$

$$\begin{bmatrix} 0.5-0.2x & 0.5+0.2x \end{bmatrix} = \begin{bmatrix} x & 1-x \end{bmatrix}$$

On comparing with L.H.S we obtain the following two equations.

0.5 – 0.2x = x	0.5 + 0.2x = 1- x
0.5 = 1.2 x	1.2x = 0.5

$$x = \frac{0.5}{1.2}$$

$$= 0.4166$$

Thus S = $\begin{bmatrix} x & 1-x \end{bmatrix}$

$$= \begin{bmatrix} 0.4166 & 1-0.4166 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4166 & 0.5834 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix}$

Suppose probability vector

$$S = \begin{bmatrix} x & 1-x \end{bmatrix} \; s.t. \; SB = S$$

$$\begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} x & 1-x \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}x + \frac{2}{3}(1-x) & \frac{1}{2}x + \frac{1}{3}(1-x) \end{bmatrix} = \begin{bmatrix} x & 1-x \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}x + \frac{2}{3} - \frac{2}{3}x & \frac{1}{2}x + \frac{1}{3} - \frac{1}{3}x \end{bmatrix} = \begin{bmatrix} x & 1-x \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{3} - \frac{1}{6}x & \frac{1}{6}x + \frac{1}{3} \end{bmatrix} = \begin{bmatrix} x & 1-x \end{bmatrix}$$

$\frac{2}{3} - \frac{1}{6}x = x$	$\frac{1}{6}x + \frac{1}{3} = 1-x$
On solving	on solving
x = 4/7	x = 4/7

Thus S = $\begin{bmatrix} x & 1-x \end{bmatrix}$

$$= \begin{bmatrix} 4/7 & 1-4/7 \end{bmatrix}$$

$$= \begin{bmatrix} 4/7 & 3/7 \end{bmatrix}$$

Example 8.8 Check the following stochastic matrix is regular or not

$$A=\begin{bmatrix}0 & 1 & 0\\ 1/5 & 2/5 & 2/5\\ 1/3 & 1/3 & 1/3\end{bmatrix}$$

Solution

We know that for any stochastic matrix to be regular, all the entries or elements of some power P^k of P should be positive for the given matrix

$$A^2=\begin{bmatrix}0 & 1 & 0\\ 1/5 & 2/5 & 2/5\\ 1/3 & 1/3 & 1/3\end{bmatrix}\times\begin{bmatrix}0 & 1 & 0\\ 1/5 & 2/5 & 2/5\\ 1/3 & 1/3 & 1/3\end{bmatrix}$$

$$=\begin{bmatrix}0\times0+1\times1/5+0\times1/3 & 0\times1+1\times2/5+0\times1/3 & 0\times0+1\times2/5+0\times1/3\\ 1/5\times0+2/5\times1/5+2/5\times1/3 & 1/5\times1+2/5\times2/5+2/5\times1/3 & 1/5\times0+2/5\times2/5+2/5\times1\\ 1/3\times0+1/3\times1/5+1/3\times1/3 & 1/3\times1+1/3\times2/5+1/3\times1/3 & 1/3\times0+1/3\times2/5+1/3\times1\end{bmatrix}$$

$$=\begin{bmatrix}1/5 & 2/5 & 2/5\\ 16/75 & 37/75 & 22/75\\ 24/135 & 26/45 & 33/135\end{bmatrix}$$

$\Rightarrow$ A is regular since, all entries in A^2 are positive.

Example 8.9 Find the unique fixed probability vector S of the following regular stochastic matrix

$$P=\begin{bmatrix}0 & 1 & 0\\ 1/5 & 2/5 & 2/5\\ 0 & 1/2 & 1/2\end{bmatrix}$$

Solution **Method I**

Suppose unique fixed vector u

$u=[x \;\; y \;\; z]$ of P

Form the matrix equation

$$[x \;\; y \;\; z]\begin{bmatrix}0 & 1 & 0\\ 1/5 & 2/5 & 2/5\\ 0 & 1/2 & 1/2\end{bmatrix}=[x \;\; y \;\; z]$$

$$\left[0\times x+\frac{1}{5}\times y+0\times 2 \quad x\times1+\frac{2}{5}\times y+\frac{1}{2}\times 2 \quad 0\times x+\frac{2}{5}y+\frac{1}{2}\times 2\right]=[x \;\; y \;\; z]$$

$$\left[\frac{1}{5}y \quad x+\frac{2}{5}y+\frac{1}{2}z \quad \frac{2}{5}y+\frac{1}{2}\right]=[x \;\; y \;\; z]$$

$$\frac{1}{5}y = x \qquad (1)$$

$$x + \frac{2}{5}y + \frac{1}{2}z = y \qquad (2)$$

$$\frac{2}{5}y + \frac{1}{2}z = z \qquad (3)$$

On solving equation (3)

$$y = \frac{5}{4}z$$

We know that the system has a non zero solution. To assign arbitrary non zero value to one of the unknowns x = 1

From equation (1) y = 5

From equation (3) z = 4

Thus u = $[1 \quad 5 \quad 4]$ is a fixed point of P.

1 + 5 + 4 = 10 then unique fixed probability vector of P is

$$S = [1/10 \quad 5/10 \quad 4/10] = [1/10 \quad 1/2 \quad 2/5]$$

Method II

If the probability vector is S s.t SP = S, it can be represented in the form $[x \quad y \quad 1-x-y]$ From the following matrix equation

$$[x \quad y \quad 1-x-y]\begin{bmatrix} 0 & 1 & 0 \\ 1/5 & 2/5 & 2/5 \\ 0 & 1/2 & 1/2 \end{bmatrix} = [x \quad y \quad 1-x-y]$$

$$\left[x \times 0 + y \times \frac{1}{5} + (1-x-y) \times 0 \quad x \times 1 + y \times \frac{2}{5} + (1-x-y)1/2 \quad x \times 0 + \frac{2}{5}y + (1-x-y)\frac{1}{2}\right]$$
$$= [x \quad y \quad 1-x-y]$$

$$\left[\frac{1}{5}y \quad x + \frac{2}{5}y + \frac{1}{2} - \frac{1}{2}x - \frac{1}{2}y. \quad \frac{2}{5}y + \frac{1}{2} - \frac{1}{2}x - \frac{1}{2}y\right] = [x \quad y \quad x-y]$$

$$\frac{1}{5}y = x \qquad\qquad \frac{1}{2}x - \frac{1}{10}y + \frac{1}{2} = \text{y}$$

y = 5x

Keeping y = 5x

x = 1/10

y = 1/2

Probability vector

$$S = [x \quad y \quad 1-x-y]$$

$$= \left[\frac{1}{10} \quad \frac{1}{2} \quad 1-\frac{1}{10}-\frac{1}{2}\right]$$

$$= \left[\frac{1}{10} \quad \frac{1}{2} \quad \frac{2}{5}\right]$$

Example 8.10 Suppose $e_0 = [1/2 \quad 1/2]$ is the initial state distribution for a markov process with the following transition matrix.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

Find e_1, e_2 and e_3

Solution

We know that

$$e_1 = e_0\ p$$

$$= [1/2 \quad 1/2] \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

$$= [1/2 \times 1/2 + 1/2 \times 1 \quad 1/2 \times 1/2 + 1/2 \times 0]$$

$$= \left[\frac{1}{4} \times \frac{1}{2} \quad \frac{1}{4} \times 0\right]$$

$$= [3/4 \quad 1/4]$$

$$e_2 = e_1\ p$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

$$= \left[\frac{3}{4} \times \frac{1}{2} \times \frac{1}{4} \times 1 \quad \frac{3}{4} \times \frac{1}{2} + \frac{1}{4} \times 0\right]$$

$$= \left[\frac{3}{8} + \frac{1}{4} \quad \frac{3}{8} + 0\right]$$

$$e_2 = \left[\frac{5}{8} \quad \frac{3}{8}\right]$$

$$e_3 = e_2\ p$$

Example 8.11 F
which are given

Solution

$P =$

Example 8.12
given below.

(a)

$$= \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

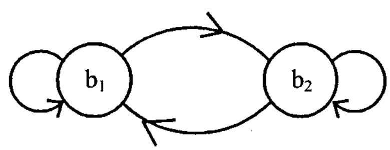
$$= \begin{bmatrix} \frac{5}{8} \times \frac{1}{2} + \frac{3}{8} \times 1 & \frac{5}{8} \times \frac{1}{2} + \frac{3}{8} \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{16} + \frac{3}{8} & \frac{5}{16} + 0 \end{bmatrix}$$

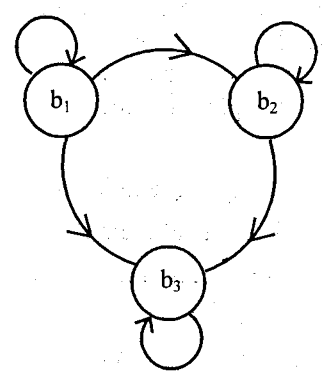
tion for a

$$e_3 = \begin{bmatrix} \frac{11}{16} & \frac{5}{16} \end{bmatrix}$$

Example 8.11 Find the transition matrix corresponding to each transition diagram which are given below.



(Diagram a)



(Diagram b)

Solution

$$P = \begin{array}{c} \\ b_1 \\ b_2 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

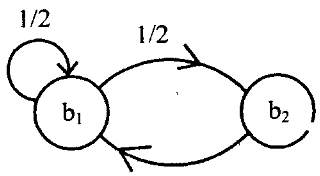
$$P = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example 8.12 Draw a transition diagram for each transition matrix which are given below.

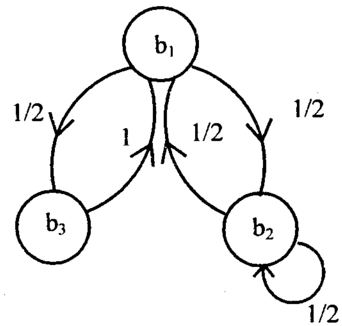
(a) $$P = \begin{array}{c} \\ b_1 \\ b_2 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \end{array}$$

(b) $$P = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Solution:



(Diagram a)



(Diagram b)

Example 8.13 A person A goes to Tirupathi temple Padmavathi temple & Shrikalahasthi. A always goes first Tirupathi then Padmavathi temple then Shrikalahasthi. If A first goes to Shrikalahasthi he is just as likely to go Tirupathi & padmavathi temple. Find the transition matrix of this markov process

Solution

Tirupathi temple $\rightarrow$ T
Padmavathi temple $\rightarrow$ P
Shrikalahasthi $\rightarrow$ S

$$T = \begin{array}{c} \\ T \\ P \\ S \end{array} \begin{array}{c} \begin{matrix} T & P & S \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Example: 8.14 A student tries to take admission in only three college A, B and C. first he goes to college A, second day to college B then third day to C. He never goes to some college in two continuous days. But if he goes either B or C, then the next day he is twice as likely to go to college A. Find out how often, in the long run he tries to each college

Solution

The transition matrix of Markov process is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

Suppose fixed vector $u = \begin{bmatrix} x & y & z \end{bmatrix}$ of transition matrix P. Thus

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \text{ or}$$

$$\frac{2}{3}y + \frac{2}{3}z = x \qquad (1)$$

$$x + \frac{1}{3}z = y \qquad (2)$$

$$\frac{1}{3}y = z \qquad (3)$$

We know that the system has a non zero solution. To assign arbitrary non zero value to one of the unknowns.

$z = 1$

From equation third $\quad y = 3$

From equation first $\frac{2}{3}\times 3 + \frac{2}{3}\times 1 = x$

$$\frac{6+2}{3} = x$$

$$\frac{8}{3} = x$$

Thus $u = \begin{bmatrix} x & y & z \end{bmatrix}$

$$= \begin{bmatrix} \frac{8}{3} & 3 & 1 \end{bmatrix} \text{ is a fixed point of P.}$$

$\frac{8}{3} + 3 + 1 = \frac{20}{3}$ Then unique fixed probability vector of P is

$$S = \begin{bmatrix} \frac{8}{3} / \frac{20}{3} & \frac{3}{1} / \frac{20}{3} & \frac{1}{1} / \frac{20}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{\cancel{3}} \times \frac{\cancel{3}}{20} & \frac{3}{1} \times \frac{3}{20} & \frac{1}{1} \times \frac{3}{20} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & \frac{9}{20} & \frac{3}{20} \end{bmatrix} = \begin{bmatrix} 0.40 & 0.45 & 0.15 \end{bmatrix}$$

Thus, in the long run, he (student) tries 40% of the time in college A, 45% of the time in college B and 15% of the time in college C.

Example 8.15 A representative from book publishing company, distributes books in only two engineering colleges A and B. First he goes to college A and second he goes to college B. He never goes to same college in two continuous days. If he goes to college B then the next day he is just as likely to go again as he is to go to college A. Find out how often, in the long run he goes to each college.

Solution

The transition matrix of the Markov process is

$$P = \begin{array}{c} \\ A \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

Suppose fixed vector $u = [x \quad y]$ of transition matrix P. Thus

$$[x \quad y] \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = [x \quad y]$$

$$\frac{1}{2}y = x \qquad (1)$$

$$x + \frac{1}{2}y = y \qquad (2)$$

We know that the system has non zero solution. To assign arbitrary non zero value to one of the unknowns x = 1

From equation (1) y = 2

Thus $u = [x \quad y]$

$$u = [1 \quad 2]$$

1 + 2 = 3 then unique fixed probability vector of P is

$S = [1/3 \quad 2/3]$

Thus in the long run, he (representative) goes 1/3 of the time to college A and 2/3 of the time to college B.

EXERCISE

Stochastic Process

1) Is the nonzero vector e= [2, 6, 0, 1] is a probability vector?

[**Hint**: All entries are nonnegative but addition of all entries=2+6+0+1=9$\neq$1
Given non zero vector is not a probability vector **Ans.**]

2) For the nonzero vector e= [2, 6, 0, 1] find probability vector

[**Hint**: v=1/9 ×e= [2/9, 6/9, 0/9, 1/9] **Ans.**]

3) If $A = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ is a square stochastic matrix and e=[3/4 1/4] is a probability vector then show that eA is also a probability vector

[**Hint**: eA= [13/16 3/16] since the sum of the elements of (eA) matrix is 13/16+3/16=1, Thus eA is a probability vector **Ans.**]

4) Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ & u= [1 2 3] Find u × A.

[**Hint**: u×A= [1 2 3]× $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

= [1×1+2×3+3×1 1×1+2×2+3×1 1×1+2×-2+3×1]
= [1+6+3 1+4+3 1-4+3]
= [10 8 0] **Ans.**]

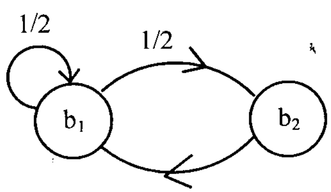
5) Check the following stochastic matrix is regular or not.

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

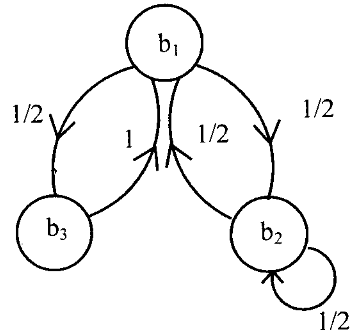
[**Hint**: $A^2 = A \times A = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{16} & \frac{13}{16} \end{bmatrix}$

A is regular since, all entries in A^2 are positive **Ans.**]

6) Find the transition matrix correspondingly to transition diagram which is given below.



(Diagram a)



(Diagram b)

[**Hint**: a) $P = \begin{array}{c} \\ b_1 \\ b_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \end{array}$ **Ans.]**

OBJECTIVE TYPE QUESTIONS

1. In the following non zero vector which is probability vector
   a) e = [2, 5, 3, 0] b) e = [3, 1, 0, 5].
   c) e = [3/9, 1/9, 0, 5/9] d) None of the above

Ans. (c)

2. Which of the following of stochastic matrix is regular matrix?
   a) $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 \\ 7/8 & 1/8 \end{bmatrix}$ d) None of the above

Ans. (a)

3. Which of the following of stochastic matrix is not regular matrix?
   a) $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 \\ 3/4 & 1/4 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 \\ 3/4 & 3/4 \end{bmatrix}$ d) None of the above

Ans. (c)

4. Which of

a) $\begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$

Ans. (d)

5. Which of

a) $\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

Ans. (c)

6. Which of

a) $\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

Ans. (c)

7. in the abo
   a) (a) is se
   c) (a) is nc

Ans. (c)

8. In the follo

a) [3/9 1

c) [1/2 1

Ans. (c)

9. In markov
   a) Equal to
   c) Less tha

Ans. (a)

4. Which of the following is stochastic matrix?

a) $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix}$ d) None of the above.

Ans. (d)

5. Which of the following is not a stochastic matrix?

a) $\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1/2 & 1/2 & 0 \end{bmatrix}$ d) None of the above

Ans. (c)

6. Which of the following is stochastic matrix?

a) $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ d) None of the above

Ans. (c)

7. in the above question (a) is not a stochastic matrix since

a) (a) is sequence matrix
b) (a) is having only two rows
c) (a) is not a square matrix
d) None of the above

Ans. (c)

8. In the following non zero vectors which is not a probability vector

a) $[3/9 \quad 1/9 \quad 0 \quad 5/9]$
b) $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$
c) $[1/2 \quad 1/3 \quad 1/6 \quad 1/6]$
d) None of the above.

Ans. (c)

9. In markov analysis, sum of the transition probabilities must be equal to

a) Equal to 1
b) greater than one
c) Less than one
d) none of the above

Ans. (a)

10. The transition matrix P of a Markov process is a

    a) Stochastic matrix   b) Always not stochastic matrix

    c) Probability vector   d) None of the above

Ans. (a)

11. In the transition matrix of the Markov chain if $P_{ij} = 0$, it denotes

    a) System is moving from j to i   b) System is moving from i to j

    c) Transition is possible   d) Transition is impossible

Ans. (d)

12. In the transition matrix of the Markov chain if $P_{ij} = 1$, it denotes

    a) System is moving from j to i   b) Both (a) & (c) one correct

    c) System is moving from i to j   d) Transition is impossible.

Ans. (c)

13. In a matrix of transition probability

    a) The element P_{ij} denotes the probability of the system is moving from i to j state or j to i state.

    b) $P_{ij} = 0$, denotes that transition is possible

    c) The sum of probability of each row is equal to one

    d) All of above are correct.

Ans. (c)

14. In a matrix of transition probability, the sum of the probability values must be equal to one, in each

    a) Row   b) Column

    c) Row & column both   d) None of the above

Ans. (a)

15. Which of the following is correct?

    a) Transition probabilities can also be represented by a probability tree diagram.

    b) The transition matrix P of a markov Process in not a stochastic matrix

    c) In Mark or analysis, sum of the transition probabilities must be equal to zero

    d) All of the above incorrect.

Ans. (a)

APPENDIX

Statistical Tables

1. Binomial distribution Function

   [Binomial probability sums: $\sum_{x=0}^{r} b(x;n,p) = \sum_{x=0}^{r} n_{c_x}\, p^x\, q^{n-x}$]

2. Poisson Distribution Function

   [Poisson probability sums: $F(x:\lambda) = \sum_{k=0}^{x} e^{-\lambda} \frac{\lambda^k}{k!}$]

3. Areas under the standard Normal

   curve from 0 to z (Normal Tables)

4. t_α – Critical values of the t-distribution

5. ${\chi_\alpha}^2$ – Critical Values of the chi-squared Distribution

6. Critical Values of the F-Distribution

   Values of $F_{0.05}(v_1, v_2)$

   Values of $F_{0.01}(v_1, v_2)$

7. Fisher's Z-Transformation

   (Values of $Z = \frac{1}{2} \ln \frac{1+r}{1-r}$)

8. Reference table of critical Values for a given L.O.S α

2. Binomial distribution Function

Binomial probability sums: $\sum_{x=0}^{r} b(x;n,p) = \sum_{x=0}^{r} n_{C_x}\, p^x q^{n-x}$

n	r	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
1	0	0.9000	0.8000	0.7500	0.7000	0.8000	0.5000	0.4000	0.3000	0.2000	0.1000
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0	0.8100	0.6400	0.5625	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100
	1	0.9900	0.9600	0.9375	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	0	0.7290	0.5120	0.4219	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010
	1	0.9720	0.8960	0.8438	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280
	2	0.9990	0.9920	0.9844	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0	0.6561	0.4096	0.3164	0.2401	0.1296	0.0625	0.0256	0.0256	0.0016	0.0001
	1	0.9477	0.8192	0.7383	0.6517	0.4752	0.3125	0.3125	0.0837	0.0272	0.0037
	2	0.9963	0.9728	0.9492	0.9163	0.8208	0.6875	0.6875	0.3483	0.1808	0.0523
	3	0.0000	0.9984	0.9961	0.9919	0.9744	0.9375	0.9375	0.7599	0.5904	0.3429
	4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0	0.5905	0.3277	0.2377	0.1681	0.0778	0.0312	0.0102	0.0024	0.0003	0.0000
	1	0.9185	0.7373	0.6328	0.5282	0.5282	0.1875	0.0870	0.0308	0.0067	0.0005
	2	0.9914	0.9421	0.8965	0.8369	0.8369	0.5000	0.3174	0.1631	0.0579	0.0086
	3	0.9995	0.9933	0.9844	0.9692	0.9692	0.8125	0.6630	0.4718	0.2627	0.0815
	4	1.0000	0.9997	0.9990	0.9976	0.9976	0.9688	0.9222	0.8319	0.6723	0.4095
	5		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	0	0.5314	0.2621	0.1780	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000
	1	0.8857	0.6554	0.5339	0.4202	0.23333	0.1094	0.0410	0.0109	0.0016	0.0001
	2	0.9841	0.9011	0.8306	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013
	3	0.9987	0.9830	0.9624	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0158
	4	0.0000	0.9984	0.9954	0.9891	0.9590	0.8906	0.7667	0.5798	0.3447	0.1143
	5	1.0000	0.9998	0.9998	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686
	6		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	0	0.4783	0.2097	0.1335	0.0824	0.0280	0.0078	0.0016	0.0002	0.0000	
	1	0.8503	0.5767	0.4449	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000
	2	0.9743	0.8520	0.7564	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002
	3	0.9973	0.9667	0.9294	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027
	4	0.9998	0.9953	0.9871	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257
	5	1.0000	0.9996	0.9987	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497
	6		1.0000	0.9999	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217
	7			1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n	r	0.10	0
8	0	0.4305	0.
	1	0.8131	0.
	2	0.9619	0.
	3	0.9950	0.
	4	0.9996	0.
	5	1.0000	0.
	6		0.
	7		1.
	8		
9	0	0.3874	0.
	1	0.7748	0.
	2	0.9470	0.
	3	0.9917	0.
	4	0.9991	0.
	5	0.9999	0.
	6	1.0000	0.
	7		1.
	8		
	9		
10	0	0.3487	0.
	1	0.7361	0.
	2	0.9298	0.
	3	0.9872	0.
	4	0.9984	0.
	5	0.9999	0.
	6	1.0000	0.
	7		0.
	8		1.
	9		
	10		
11	0	0.3138	0.
	1	0.6974	0.
	2	0.9104	0.
	3	0.9815	0.
	4	0.9972	0.
	5	0.9997	0.
	6	1.0000	0.
	7		0.
	8		1.
	9		
	10		
	11		

)	0.90
)0	0.1000
)0	1.0000
00	0.0100
00	0.1900
00	1.0000
80	0.0010
40	0.0280
80	0.2710
00	1.0000
16	0.0001
72	0.0037
08	0.0523
04	0.3429
00	1.0000
03	0.0000
067	0.0005
579	0.0086
627	0.0815
723	0.4095
000	1.0000
001	0.0000
016	0.0001
170	0.0013
989	0.0158
447	0.1143
379	0.4686
000	1.0000
000	
004	0.0000
047	0.0002
333	0.0027
1480	0.0257
4233	0.1497
7903	0.5217
0000	1.0000

n	r	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
8	0	0.4305	0.1678	0.1001	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	
	1	0.8131	0.5033	0.3671	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	
	2	0.9619	0.7969	0.6785	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000
	3	0.9950	0.9437	0.8862	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004
	4	0.9996	0.9896	0.9727	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050
	5	1.0000	0.9988	0.9958	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381
	6		0.9991	0.9996	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869
	7		1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695
	8				1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	0	0.3874	0.1342	0.0751	0.0404	0.0101	0.0020	0.0003	0.0000		
	1	0.7748	0.4362	0.3003	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	
	2	0.9470	0.7382	0.6007	0.4628	0.2318	0.0898	0.0250	0.0043	0.0003	0.0000
	3	0.9917	0.9144	0.8343	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001
	4	0.9991	0.9804	0.9511	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009
	5	0.9999	0.9969	0.9900	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083
	6	1.0000	0.9997	0.9987	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530
	7		1.0000	0.9999	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252
	8			1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126
	9					1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0	0.3487	0.1074	0.0563	0.0282	0.0060	0.0010	0.0001	0.0000		
	1	0.7361	0.3758	0.2440	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	
	2	0.9298	0.6778	0.5256	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	
	3	0.9872	0.8791	0.7759	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000
	4	0.9984	0.9672	0.9219	0.8497	0.6331	0.3770	0.1662	0.0474	0.0064	0.0002
	5	0.9999	0.9936	0.9803	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016
	6	1.0000	0.9991	0.9965	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128
	7		0.9999	0.9996	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702
	8		1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639
	9				1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513
	10					1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	0	0.3138	0.0859	0.0422	0.0198	0.0036	0.0005	0.0000			
	1	0.6974	0.3221	0.1971	0.1130	0.0302	0.0059	0.0007	0.0000		
	2	0.9104	0.6174	0.4552	0.3127	0.1189	0.0327	0.0059	0.0006	0.0000	
	3	0.9815	0.8369	0.7133	0.5696	0.2963	0.1133	0.0293	0.0043	0.0002	
	4	0.9972	0.9496	0.8854	0.7897	0.5328	0.2744	0.0994	0.0216	0.0020	0.0000
	5	0.9997	0.9883	0.9657	0.9218	0.7535	0.5000	0.2465	0.0782	0.0117	0.0003
	6	1.0000	0.9980	0.9924	0.9784	0.9006	0.7256	0.4672	0.2103	0.0504	0.0028
	7		0.9998	0.9988	0.9957	0.9707	0.8867	0.7037	0.4304	0.1611	0.0185
	8		1.0000	0.9999	0.9994	0.9941	0.9673	0.8811	0.6873	0.3826	0.0896
	9			1.0000	1.0000	0.9993	0.9941	0.9698	0.8870	0.6779	0.3026
	10					1.0000	0.9995	0.9964	0.9802	0.9141	0.6862
	11						1.0000	1.0000	1.0000	1.0000	1.0000

n	r	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
12	0	0.2824	0.0687	0.0317	0.0138	0.0022	0.0002	0.0000			
	1	0.6590	0.2749	0.1584	0.0850	0.0196	0.0032	0.0003	0.0000		
	2	0.8891	0.5583	0.3907	0.2528	0.0834	0.0193	0.0028	0.0002	0.0000	
	3	0.9744	0.7946	0.6488	0.4925	0.2253	0.0730	0.0153	0.0017	0.0001	
	4	0.9957	0.9274	0.8424	0.7237	0.4382	0.1938	0.0573	0.0095	0.0006	0.0000
	5	0.9995	0.9806	0.9456	0.8821	0.6652	0.3872	0.1582	0.0386	0.0039	0.0001
	6	0.9999	0.9961	0.9857	0.9614	0.8418	0.6128	0.3348	0.1178	0.0194	0.0005
	7	1.0000	0.9994	0.9972	0.9905	0.9427	0.8062	0.5618	0.2763	0.0726	0.0043
	8		0.9999	0.9996	0.9983	0.9847	0.9270	0.7747	0.5075	0.2054	0.0256
	9		1.0000	1.0000	0.9998	0.9972	0.9807	0.9166	0.7472	0.4417	0.1109
	10				1.0000	0.9997	0.9968	0.9804	0.9150	0.7251	0.3410
	11					1.0000	0.9998	0.9978	0.9862	0.9313	0.7176
	12						1.0000	1.0000	1.0000	1.0000	1.0000
13	0	0.2542	0.0550	0.0238	0.0097	0.0013	0.0001	0.0000			
	1	0.6213	0.2336	0.1267	0.0637	0.0126	0.0017	0.0001	0.0000		
	2	0.8661	0.5017	0.3326	0.2025	0.0579	0.0112	0.0013	0.0001		
	3	0.9658	0.7473	0.5843	0.4206	0.1686	0.0461	0.0078	0.0007	0.0000	
	4	0.9935	0.9009	0.7940	0.6543	0.3530	0.1334	0.0321	0.0040	0.0002	
	5	0.9991	0.9700	0.9198	0.8346	0.5744	0.2905	0.0977	0.0182	0.0012	0.0000
	6	0.9999	0.9930	0.9757	0.9376	0.7712	0.5000	0.2288	0.0624	0.0070	0.0001
	7	1.0000	0.9980	0.9944	0.9818	0.9023	0.7095	0.4256	0.1654	0.0300	0.0009
	8		0.9998	0.9990	0.9960	0.9679	0.8666	06470	0.3457	0.0991	0.0065
	9		1.0000	0.9999	0.9993	0.9922	0.9539	0.8314	0.5794	0.2527	0.0342
	10			1.0000	0.9999	0.9987	0.9888	0.9421	0.7975	0.4983	0.1339
	11				1.0000	0.9999	0.9983	0.9874	0.9363	0.7664	0.3787
	12					1.0000	0.9999	0.9987	0.9903	0.9450	0.7458
	13						1.0000	1.0000	1.0000	1.0000	1.0000
14	0	0.2288	0.0440	0.0178	0.0068	0.0008	0.0001	0.0000			
	1	0.5846	0.1979	0.1010	0.0475	0.0081	0.0009	0.0001			
	2	0.8416	0.4481	0.2811	0.1608	0.0398	0.0065	0.0006	0.0000		
	3	0.9559	0.6982	0.5213	0.3552	0.1243	0.0287	0.0039	0.0002		
	4	0.9908	0.8702	0.7415	0.5842	0.2793	0.0898	0.0175	0.0017	0.0000	
	5	0.9985	0.9561	0.8883	0.7805	0.4859	0.2120	0.0583	0.0083	0.0004	
	6	0.9998	0.9884	0.9617	0.9067	0.6925	0.3953	0.1501	0.0315	0.0024	0.0000
	7	1.0000	0.9976	0.9897	0.9685	0.8499	0.6047	0.3075	0.0933	0.0116	0.0002
	8		0.9996	0.9978	0.9917	0.9417	0.7880	0.5141	0.2195	0.0439	0.0015
	9		1.0000	0.9997	0.9983	0.9825	0.9102	0.7207	0.4158	0.1298	0.0092
	10			1.0000	0.9998	0.9961	0.9713	0.8757	0.6448	0.3018	0.0441
	11				1.0000	0.9994	0.9935	0.9602	0.8392	0.5519	0.1584
	12					0.9999	0.9991	0.9919	0.9525	0.8021	0.4154
	13					1.0000	0.9999	0.9992	0.9932	0.9560	0.7712
	14						1.0000	1.0000	1.0000	1.0000	1.0000

n	r	0.10
15	0	0.2059
	1	0.5490
	2	0.8159
	3	0.9444
	4	0.9873
	5	0.9978
	6	0.9997
	7	1.0000
	8	
	9	
	10	
	11	
	12	
	13	
	14	
	15	
16	0	0.1853
	1	0.5147
	2	0.7892
	3	0.9316
	4	0.9830
	5	0.9967
	6	0.9995
	7	0.9999
	8	1.0000
	9	
	10	
	11	
	12	
	13	
	14	
	15	
	16	
17	0	0.1668
	1	0.4818
	2	0.7618
	3	0.9174
	4	0.9779
	5	0.9953
	6	0.9992
	7	0.9999
	8	1.0000
	9	
	10	
	11	
	12	
	13	
	14	
	15	
	16	
	17	

.80	0.90
)000	
)001	
)006	0.0000
)039	0.0001
)194	0.0005
)726	0.0043
205⁄	0.0256
4417	0.1109
7251	0.3410
9313	0.7176
0000	1.0000
0000	
0002	
0012	0.0000
0070	0.0001
0300	0.0009
0991	0.0065
2527	0.0342
4983	0.1339
7664	0.3787
9450	0.7458
0000	1.0000
0000	
0004	
0024	0.0000
0116	0.0002
0439	0.0015
1?98	0.0092
3018	0.0441
5519	0.1584
8021	0.4154
9560	0.7712
0000	1.0000

n	r	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
15	0	0.2059	0.0352	0.0134	0.0047	0.0005	0.0000				
	1	0.5490	0.1671	0.0802	0.0353	0.0052	0.0005	0.0000			
	2	0.8159	0.3980	0.2361	0.1268	0.0271	0.0037	0.0003	0.0000		
	3	0.9444	0.6482	0.4613	0.2969	0.0905	0.0176	0.0019	0.0001		
	4	0.9873	0.8358	0.6865	0.5155	0.2173	0.0592	0.0094	0.0007	0.0000	
	5	0.9978	0.9389	0.8516	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	
	6	0.9997	0.9819	0.9434	0.8689	0.6098	0.3036	0.0951	0.0152	0.0008	
	7	1.0000	0.9958	0.9827	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000
	8		0.9992	0.9958	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003
	9		0.9999	0.9992	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0023
	10		1.0000	0.9999	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127
	11			1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556
	12				1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841
	13					1.0000	0.9995	0.9948	0.9647	0.8329	0.4510
	14						1.0000	0.9995	0.9953	0.9648	0.7941
	15							1.0000	1.0000	1.0000	1.0000
16	0	0.1853	0.0281	0.0100	0.0033	0.0003	0.0000				
	1	0.5147	0.1407	0.0635	0.0261	0.0033	0.0003	0.0000			
	2	0.7892	0.3518	0.1971	0.0994	0.0183	0.0021	0.0001			
	3	0.9316	0.5981	0.4050	0.2459	0.0651	0.0106	0.0009	0.0000		
	4	0.9830	0.7982	0.6302	0.4499	0.1666	0.0384	0.0049	0.0003		
	5	0.9967	0.9183	0.8103	0.6598	0.3288	0.1051	0.0191	0.0016	0.0000	
	6	0.9995	0.9733	0.9204	0.8247	0.5272	0.2272	0.0583	0.0071	0.0002	
	7	0.9999	0.9930	0.9729	0.9256	0.7161	0.4018	0.1423	0.0257	0.0015	0.0000
	8	1.0000	0.9985	0.9925	0.9743	0.8577	0.5982	0.2839	0.0744	0.0070	0.0001
	9		0.9998	0.9984	0.9929	0.9417	0.7728	0.4728	0.1753	0.0267	0.0005
	10		1.0000	0.9997	0.9984	0.9809	0.8949	0.6712	0.3402	0.0817	0.0033
	11			1.0000	0.9997	0.9951	0.9616	0.8334	0.5501	0.2018	0.0170
	12				1.0000	0.9991	0.9894	0.9349	0.7541	0.4019	0.0684
	13					0.9999	0.9979	0.9817	0.9006	0.6482	0.2108
	14					1.0000	0.9997	0.9967	0.9739	0.8593	0.4853
	15						1.0000	0.9997	0.9967	0.9719	0.8147
	16							1.0000	1.0000	1.0000	1.0000
17	0	0.1668	0.0225	0.0075	0.0023	0.0002	0.0000				
	1	0.4818	0.1182	0.0501	0.0193	0.0021	0.0001	0.0000			
	2	0.7618	0.3096	0.1637	0.0774	0.0123	0.0012	0.0001			
	3	0.9174	0.5489	0.3530	0.2019	0.0464	0.0064	0.0005	0.0000		
	4	0.9779	0.7582	0.5739	0.3887	0.1260	0.0245	0.0025	0.0001		
	5	0.9953	0.8943	0.7653	0.5968	0.2639	0.0717	0.0106	0.0007	0.0000	
	6	0.9992	0.9623	0.8929	0.7752	0.4478	0.1662	0.0348	0.0032	0.0001	
	7	0.9999	0.9891	0.9598	0.8954	0.6405	0.3145	0.0919	0.0127	0.0005	
	8	1.0000	0.9974	0.9876	0.9597	0.8011	0.5000	0.1989	0.0403	0.0026	0.0000
	9		0.9995	0.9969	0.9873	0.9081	0.6855	0.3595	0.1046	0.0109	0.0001
	10		0.9999	0.9994	0.9968	0.9652	0.8338	0.5522	0.2248	0.0377	0.0008
	11		1.0000	0.9999	0.9993	0.9894	0.9283	0.7361	0.4032	0.1057	0.0047
	12			1.0000	0.9999	0.9975	0.9755	0.8740	0.6113	0.2418	0.0221
	13				1.0000	0.9995	0.9936	0.9536	0.7981	0.4511	0.0826
	14					0.9999	0.9988	0.9877	0.9226	0.6904	0.2382
	15					1.0000	0.9999	0.9979	0.9807	0.8818	0.5182
	16						1.0000	0.9998	0.9977	0.9775	0.8332
	17						1.0000	1.0000	1.0000	1.0000	1.0000

n	r	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
18	0	0.1501	0.0180	0.0056	0.0016	0.0001	0.0000				
	1	0.4503	0.0991	0.0395	0.0142	0.0013	0.0001				
	2	0.7338	0.2713	0.1353	0.0600	0.0082	0.0007	0.0000			
	3	0.9018	0.5010	0.3057	0.1646	0.0328	0.0038	0.0002			
	4	0.9718	0.7164	0.5787	0.3327	0.0942	0.0154	0.0013	0.0000		
	5	0.9936	0.8671	0.7175	0.5344	0.2088	0.0481	0.0058	0.0003		
	6	0.9988	0.9487	0.8610	0.7217	0.3743	0.1189	0.0203	0.0014	0.0000	
	7	0.9998	0.9837	0.9431	0.8593	0.5634	0.2403	0.0576	0.0061	0.0002	
	8	1.0000	0.9957	0.9807	0.9404	0.7368	0.4073	0.1347	0.0210	0.0009	
	9		0.9991	0.9946	0.9790	0.8653	0.5927	0.2632	0.0596	0.0043	0.0000
	10		0.9998	0.9988	0.9939	0.9424	0.7597	0.4366	0.1407	0.0163	0.0002
	11		1.0000	0.9998	0.9986	0.9797	0.8811	0.6257	0.2783	0.0513	0.0012
	12			1.0000	0.9997	0.9942	0.9519	0.7912	0.4656	0.1329	0.0064
	13				1.0000	0.9987	0.9846	0.9058	0.6673	0.2836	0.0282
	14					0.9998	0.9962	0.9672	0.8354	0.4990	0.0982
	15					1.0000	0.9993	0.9918	0.9400	0.7287	0.2662
	16						0.9999	0.9918	0.9858	0.9009	0.5497
	17						1.0000	0.9987	0.9984	0.9820	0.8499
	18							0.9999	1.0000	1.0000	1.0000
								1.0000			
19	0	0.1351	0.0144	0.0042	0.0011	0.0001					
	1	0.4203	0.0829	0.0310	0.0104	0.0008	0.0000				
	2	0.7054	0.2369	0.1113	0.0462	0.0055	0.0004	0.0000			
	3	0.8850	0.4551	0.2631	0.1332	0.0230	0.0022	0.0001			
	4	0.9648	0.6733	0.4654	0.2822	0.0696	0.0096	0.0006	0.0000		
	5	0.9914	0.8369	0.6678	0.4739	0.1629	0.0318	0.0031	0.0001		
	6	0.9983	0.9324	0.8251	0.6655	0.3081	0.0835	0.0116	0.0006	0.0000	
	7	0.9997	0.9767	0.9225	0.8180	0.4878	0.1796	0.0352	0.0028	0.0003	
	8	1.0000	0.9933	0.9713	0.9161	0.6675	0.3238	0.0885	0.0105	0.0016	
	9		0.9984	0.9911	0.9674	0.8139	0.5000	0.1861	0.0326	0.0067	0.0000
	10		0.9997	0.9977	0.9895	0.9115	0.6762	0.3325	0.0839	0.0233	0.0003
	11		0.9999	0.9995	0.9972	0.9648	0.8204	0.5122	0.1820	0.0676	0.0017
	12		1.0000	0.9999	0.9994	0.9884	0.9165	0.6919	0.3345	0.1631	0.0086
	13			1.0000	0.9999	0.9969	0.9682	0.8371	0.5261	0.3267	0.0352
	14				1.0000	0.9994	0.9904	0.9304	0.7178	0.5449	0.1150
	15					0.9999	0.9978	0.9770	0.8668	0.7631	0.2946
	16					1.0000	0.9996	0.9945	0.9538	0.9171	0.5797
	17						1.0000	0.9992	0.9896	0.9856	0.8649
	18							0.9999	0.9989	1.0000	1.0000
	19							1.0000	1.0000		

20	0	0.1216
	1	0.3917
	2	0.6769
	3	0.8670
	4	0.9568
	5	0.9887
	6	0.9976
	7	0.9996
	8	0.9999
	9	1.0000
	10	
	11	
	12	
	13	
	14	
	15	
	16	
	17	
	18	
	19	
	20	

80	0.90
)000	
)002	
)009	
)043	0.0000
)163	0.0002
)513	0.0012
1329	0.0064
2836	0.0282
4990	0.0982
7287	0.2662
9009	0.5497
9820	0.8499
0000	1.0000
0000	
0003	
0016	
0067	0.0000
0233	0.0003
0676	0.0017
1631	0.0086
3267	0.0352
5449	0.1150
7631	0.2946
.9171	0.5797
.9856	0.8649
.0000	1.0000

20	0	0.1216	0.0115	0.0032	0.0008	0.0000					
	1	0.3917	0.0692	0.0243	0.0076	0.0005	0.0000				
	2	0.6769	0.2061	0.0913	0.0355	0.0036	0.0002	0.0000			
	3	0.8670	0.4114	0.2252	0.1071	0.0160	0.0013	0.0001			
	4	0.9568	0.6296	0.4148	0.2375	0.0510	0.0059	0.0003			
	5	0.9887	0.8042	0.6172	0.4164	0.1256	0.0207	0.0016	0.0000		
	6	0.9976	0.9133	0.7858	0.6080	0.2500	0.0577	0.0065	0.0003		
	7	0.9996	0.9679	0.8982	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	
	8	0.9999	0.9900	0.9591	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	
	9	1.0000	0.9974	0.9861	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	
	10		0.9994	0.9961	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000
	11		0.9999	0.9991	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001
	12		1.0000	0.9998	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004
	13			1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024
	14				1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113
	15					0.9997	0.9941	0.9490	0.7625	0.3704	0.432
	16					1.0000	0.9987	0.9840	0.8029	0.5886	0.1330
	17						0.9998	0.9964	0.9645	0.7939	0.3231
	18						1.0000	0.9995	0.9924	0.9308	0.6083
	19							1.0000	0.9992	0.9885	0.8784
	20								1.0000	1.0000	1.0000

2. Poisson Distribution Function

Poisson probability sums: $F(x:\lambda)=\sum_{k=0}^{x} e^{-\lambda}\frac{\lambda^k}{k!}$

λ \ x	0	1	2	3	4	5	6	7	8	9
0.02	0.980	1.000								
0.04	0.961	0.999	1.000							
0.06	0.942	0.998	1.000							
0.08	0.923	0.997	1.000							
0.10	0.905	0.995	1.000							
0.15	0.861	0.990	0.999	1.000						
0.20	0.819	0.982	0.999	1.000						
0.25	0.779	0.974	0.998	1.000						
0.30	0.741	0.963	0.996	1.000						
0.35	0.705	0.951	0.994	1.000						
0.40	0.670	0.938	0.992	0.999	1.000					
0.45	0.638	0.925	0.989	0.999	1.000					
0.50	0.607	0.910	0.986	0.998	1.000					
0.55	0.577	0.894	0.982	0.998	1.000					
0.60	0.549	0.878	0.977	0.997	1.000					
0.65	0.522	0.861	0.972	0.996	0.999	1.000				
0.70	0.497	0.844	0.966	0.994	0.999	1.000				
0.75	0.472	0.827	0.959	0.993	0.999	1.000				
0.80	0.449	0.809	0.953	0.991	0.999	1.000				
0.85	0.427	0.791	0.945	0.989	0.998	1.000				
0.90	0.407	0.772	0.937	0.987	0.998	1.000				
0.95	0.387	0.754	0.929	0.984	0.997	1.000				
1.00	0.368	0.736	0.920	0.981	0.996	0.999	1.000			
1.1	0.333	0.699	0.900	0.974	0.995	0.999	1.000			
1.2	0.301	0.663	0.879	0.966	0.992	0.998	1.000			
1.3	0.273	0.627	0.857	0.957	0.989	0.998	1.000			
1.4	0.247	0.592	0.833	0.946	0.986	0.997	0.999	1.000		
1.5	0.223	0.558	0.809	0.934	0.981	0.996	0.999	1.000		
1.6	0.202	0.525	0.783	0.921	0.976	0.994	0.999	1.000		
1.7	0.183	0.493	0.757	0.907	0.970	0.992	0.998	1.000		
1.8	0.165	0.463	0.731	0.891	0.964	0.990	0.997	0.999	1.000	
1.9	0.150	0.434	0.704	0.875	0.956	0.987	0.997	0.999	1.000	
2.0	0.135	0.406	0.677	0.857	0.947	0.983	0.995	0.999	1.000	

Poisson distribution Function: $F(x:\lambda)=\sum_{k=0}^{x} e^{-\lambda}\frac{\lambda^k}{k!}$

Poisson distribution Function: $F(x:\lambda)=\sum_{k=0}^{x} e^{-\lambda}\frac{\lambda^k}{k!}$

λ \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2.2	0.111	0.355	0.623	0.819	0.928	0.975	0.993	0.998	1.000								
2.4	0.091	0.308	0.570	0.779	0.904	0.964	0.988	0.997	0.999	1.000							
2.6	0.074	0.267	0.518	0.7.6	0.877	0.951	0.983	0.995	0.999	1.000							
2.8	0.061	0.231	0.469	0.692	0.848	0.935	0.976	0.992	0.998	0.999	1.000						
3.0	0.050	0.199	0.423	0.647	0.815	0.916	0.966	0.988	0.996	0.999	1.000						
3.2	0.041	0.171	0.380	0.603	0.781	0.895	0.955	0.983	0.994	0.998	1.000						
3.4	0.033	0.147	0.340	0.558	0.744	0.871	0.942	0.977	0.992	0.997	0.999	1.000					
3.6	0.027	0.126	0.303	0.515	0.706	0.844	0.927	0.969	0.988	0.996	0.999	1.000					
3.8	0.022	0.107	0.269	0.473	0.668	0.816	0.909	0.960	0.984	0.994	0.998	0.999	1.000				
4.0	0.018	0.092	0.238	0.433	0.629	0.785	0.889	0.949	0.979	0.992	0.997	0.999	1.000				
4.2	0.015	0.078	0.210	0.395	0.590	0.753	0.867	0.936	0.972	0.989	0.996	0.999	1.000				
4.4	0.012	0.066	0.185	0.359	0.551	0.720	0.844	0.921	0.964	0.985	0.994	0.998	0.999	1.000			
4.6	0.010	0.056	0.163	0.326	0.513	0.686	0.818	0.905	0.955	0.980	0.992	0.997	0.999	1.000			
4.8	0.008	0.048	0.143	0.294	0.476	0.651	0.791	0.887	0.944	0.975	0.990	0.996	0.999	1.000			
5.0	0.007	0.040	0.125	0.265	0.440	0.616	0.762	0.867	0.932	0.968	0.986	0.995	0.998	0.999	1.000		
5.2	0.006	0.034	0.109	0.238	0.406	0.581	0.732	0.845	0.918	0.960	0.982	0.993	0.997	0.999	1.000		
5.4	0.005	0.029	0.095	0.213	0.373	0.546	0.702	0.822	0.903	0.951	0.977	0.990	0.996	0.999	1.000		
5.6	0.004	0.024	0.082	0.191	0.342	0.512	0.670	0.797	0.886	0.941	0.972	0.988	0.995	0.998	0.999	1.000	
5.8	0.003	0.021	0.072	0.170	0.313	0.478	0.638	0.771	0.867	0.929	0.965	0.984	0.993	0.997	0.999	1.000	
6.0	0.002	0.017	0.062	0.151	0.285	0.446	0.606	0.744	0.847	0.916	0.957	0.980	0.991	0.996	0.999	0.999	1.000

Poisson distribution Function: $F(x,\lambda)=\sum_{k=0}^{x} e^{-\lambda}\frac{\lambda^k}{k!}$

λ \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
6.2	0.002	0.015	0.054	0.134	0.259	0.414	0.716	0.826	0.826	0.902	0.949	0.975	0.989	0.995	0.998	0.999	1.000						
6.4	0.002	0.012	0.046	0.119	0.235	0.384	0.687	0.803	0.803	0.886	0.939	0.969	0.986	0.994	0.997	0.999	1.000						
6.6	0.001	0.010	0.040	0.105	0.213	0.355	0.658	0.780	0.780	0.869	0.927	0.963	0.982	0.992	0.997	0.999	0.999	1.000					
6.8	0.001	0.009	0.034	0.093	0.192	0.327	0.628	0.755	0.755	0.850	0.915	0.955	0.978	0.990	0.996	0.998	0.999	1.000					
7.0	0.001	0.007	0.030	0.082	0.173	0.301	0.599	0.729	0.729	0.830	0.901	0.947	0.973	0.987	0.994	0.998	0.999	1.000					
7.2	0.001	0.006	0.025	0.072	0.156	0.276	0.569	0.703	0.703	0.810	0.887	0.937	0.967	0.984	0.993	0.997	0.999	1.000	1.000				
7.4	0.001	0.005	0.022	0.063	0.140	0.253	0.539	0.676	0.676	0.788	0.871	0.926	0.961	0.980	0.991	0.996	0.998	0.999	1.000				
7.6	0.001	0.004	0.019	0.055	0.125	0.231	0.510	0.648	0.648	0.765	0.854	0.915	0.954	0.976	0.989	0.995	0.998	0.999	1.000				
7.8	0.000	0.004	0.016	0.048	0.112	0.210	0.481	0.620	0.620	0.741	0.835	0.902	0.945	0.971	0.986	0.993	0.997	0.999	1.000				
8.0	0.000	0.003	0.014	0.042	0.100	0.191	0.453	0.593	0.593	0.717	0.816	0.888	0.936	0.966	0.983	0.992	0.996	0.998	0.999	1.000			
8.5	0.000	0.002	0.009	0.030	0.074	0.150	0.386	0.523	0.523	0.653	0.763	0.849	0.909	0.949	0.973	0.986	0.993	0.997	0.999	0.999	1.000		
9.0	0.000	0.001	0.006	0.021	0.055	0.116	0.324	0.456	0.456	0.587	0.706	0.803	0.876	0.926	0.959	0.978	0.989	0.995	0.998	0.999	1.000		
9.5	0.000	0.001	0.004	0.015	0.040	0.089	0.269	0.392	0.392	0.522	0.645	0.752	0.836	0.898	0.940	0.967	0.982	0.991	0.996	0.998	0.999	1.000	
10.0	0.000	0.000	0.003	0.010	0.029	0.067	0.220	0.333	0.333	0.458	0.583	0.697	0.792	0.864	0.917	0.951	0.973	0.986	0.993	0.997	0.999	0.999	1.000

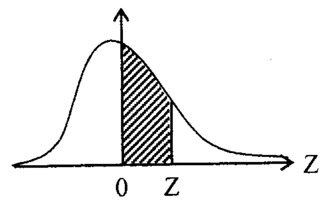
Poisson distr

λ \ x	0
10.5	0.000
11.0	0.000
11.5	0.000
12.0	0.000
12.5	0.000
13.0	0.000
13.5	0.000
14.0	0.000
14.5	0.000
15.0	0.000
	10
10.5	0.521
11.0	0.460
11.5	0.402
12.0	0.347
12.5	0.297
13.0	0.252
13.5	0.211
14.0	0.176
14.5	0.145
15.0	0.118
	20
10.5	0.997
11.0	0.995
11.5	0.992
12.0	0.988
12.5	0.983
13.0	0.975
13.5	0.965
14.0	0.952
14.5	0.936
15.0	0.917

Poisson distribution Function: $F(x:\lambda)=\sum_{k=0}^{x} e^{-\lambda}\frac{\lambda^k}{k!}$

λ \ x	0	1	2	3	4	5	6	7	8	9
10.5	0.000	0.000	0.002	0.007	0.021	0.050	0.102	0.179	0.279	0.397
11.0	0.000	0.000	0.001	0.005	0.015	0.038	0.079	0.143	0.232	0.341
11.5	0.000	0.000	0.001	0.003	0.011	0.028	0.060	0.114	0.191	0.289
12.0	0.000	0.000	0.001	0.002	0.008	0.020	0.046	0.090	0.155	0.242
12.5	0.000	0.000	0.000	0.002	0.005	0.015	0.035	0.070	0.125	0.201
13.0	0.000	0.000	0.000	0.001	0.004	0.011	0.026	0.054	0.100	0.166
13.5	0.000	0.000	0.000	0.001	0.003	0.008	0.019	0.041	0.079	0.135
14.0	0.000	0.000	0.000	0.000	0.002	0.006	0.014	0.032	0.062	0.109
14.5	0.000	0.000	0.000	0.000	0.001	0.004	0.010	0.024	0.048	0.088
15.0	0.000	0.000	0.000	0.000	0.001	0.003	0.008	0.018	0.037	0.070
	10	11	12	13	14	15	16	17	18	19
10.5	0.521	0.639	0.742	0.825	0.888	0.932	0.960	0.978	0.988	0.994
11.0	0.460	0.579	0.689	0.781	0.854	0.907	0.944	0.968	0.982	0.991
11.5	0.402	0.520	0.633	0.733	0.815	0.878	0.924	0.954	0.974	0.986
12.0	0.347	0.462	0.576	0.682	0.772	0.844	0.899	0.937	0.963	0.979
12.5	0.297	0.406	0.519	0.628	0.725	0.806	0.869	0.916	0.948	0.969
13.0	0.252	0.353	0.463	0.573	0.675	0.764	0.835	0.890	0.930	0.957
13.5	0.211	0.304	0.409	0.518	0.623	0.718	0.798	0.861	0.908	0.942
14.0	0.176	0.260	0.358	0.464	0.570	0.669	0.756	0.827	0.883	0.923
14.5	0.145	0.220	0.311	0.413	0.518	0.619	0.711	0.790	0.853	0.901
15.0	0.118	0.185	0.268	0.363	0.466	0 568	0.664	0.749	0.819	0.875
	20	21	22	23	24	25	26	27	28	29
10.5	0.997	0.999	0.999	1.000						
11.0	0.995	0.998	0.999	1.000						
11.5	0.992	0.996	0.998	0.999	1.000					
12.0	0.988	0.994	0.997	0.999	0.999	1.000				
12.5	0.983	0.991	0.995	0.998	0.999	0.999	1.000			
13.0	0.975	0.986	0.992	0.996	0.998	0.999	1.000			
13.5	0.965	0.980	0.989	0.994	0.997	0.998	0.999	1.000		
14.0	0.952	0.971	0.983	0.991	0.995	0.997	0.999	0.999	1.000	
14.5	0.936	0.960	0.976	0.986	0.992	0.996	0.998	0.999	0.999	1.000
15.0	0.917	0.947	0.967	0.981	0.989	0.994	0.997	0.998	0.999	1.000

3. Areas under the standard Normal curve from 0 to z (Normal Tables)



z	0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	1141
0.3	.1179	.1217	.1256	.1293	.1331	.1368	.1406	.1443	.1480	1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	1879
0.5	.1916	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	2649
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	4177
1.4	.4192	.4207	.4222	.4263	.4251	.4265	.4279	.4292	.4306	4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	4545
1.7	.4654	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	4974
2.8	.4974	.4975	.4976	.4977	.4979	.4978	.4979	.4979	.4980	4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	5000

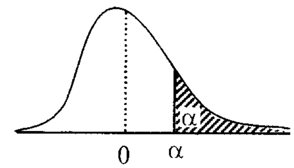
4. t_α – Crit

ν	0.40
1	0.325
2	0.289
3	0.277
4	0.271
5	0.267
6	0.265
7	0.263
8	0.262
9	0.261
10	0.260
11	0.260
12	0.259
13	0.259
14	0.258
15	0.258
16	0.258
17	0.257
18	0.257
19	0.257
20	0.257
21	0.257
22	0.256
23	0.256
24	0.256
25	0.256
26	0.256
27	0.256
28	0.256
29	0.256
30	0.256
40	0.255
60	0.254
120	0.254
∞	0.253

$\Longrightarrow$ Z

	9
)	0359
ł	0754
}	1141
)	1517
ł	1879
)	2224
3	2649
3	2852
5	3133
5	3389
9	3621
0	3830
7	4015
2	4177
6	4319
9	4441
5	4545
5	4633
9	4706
1	4767
2	4817
4	4857
7	4890
3	4916
4	4936
1	4952
3	4964
3	4974
0	4981
6	4986
0	4990
3	4993
5	4995
6	4997
7	4998
8	4998
9	4999
9	4999
9	4999
0	5000

4. t_α – Critical values of the t-distribution



ν	0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.537	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	0.293
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.318	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

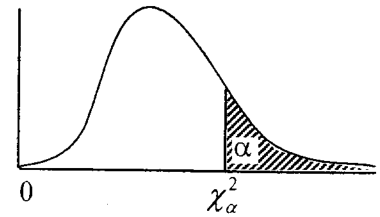
t_α – Critical Values of the t-distribution

	α						
ν	**0.02**	**0.015**	**0.01**	**0.0075**	**0.005**	**0.0025**	**0.0005**
1	15.895	21.205	31.821	42.434	63.657	127.322	636.590
2	4.849	5.643	6.965	8.073	9.925	14.089	31.598
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	6.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.849
21	2.189	2.328	2.18	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.690
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.659
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.125	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.291

5. $\chi_\alpha^{\,2}$ – Critic

ν	**0.995**
1	0.04393
2	0.0100
3	0.0717
4	0.207
5	0.412
6	0.676
7	0.989
8	1.344
9	1.735
10	2.156
11	2.603
12	3.074
13	3.565
14	4.075
15	4.601
16	5.142
17	5.697
18	6.265
19	6.844
20	7.434
21	8.034
22	8.643
23	9.260
24	9.886
25	10.520
26	11.160
27	11.808
28	12.461
29	13.121
30	13.787

0.0005
636.590
31.598
12.924
8.610
6.869
5.959
5.408
5.041
4.781
4.587
4.437
4.318
4.221
4.140
4.073
4.015
3.965
3.922
3.883
3.849
3.819
3.792
3.768
3.745
3.725
3.707
3.690
3.674
3.659
3.646
3.551
3.460
3.373
3.291



5. χ_α^2 – Critical Values of the chi-squared Distribution

ν	α									
	0.995	0.99	0.98	0.975	0.95	0.90	0.80	0.75	0.75	0.50
1	0.04393	0.03157	0.03628	0.03982	0.00393	0.0158	0.0642	0.102	0.148	0.455
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	0.575	0.713	1.386
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	1.213	1.424	2.366
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	1.923	2.195	3.357
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	2.675	3.000	4.351
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
8	1.344	1.646	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
9	1.735	2.088	0.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.937	7.267	9.342
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
13	3.565	4.107	4.765	5.009	5.892	7.042	8.634	9.299	9.926	12.340
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.036	11.721	14.339
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	11.912	12.624	15.338
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	12.792	13.531	16.338
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	13.675	14.440	17.338
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	14.562	15.352	18.338
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	15.452	16.266	19.337
21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	16.344	17.182	20.337
22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	17.240	18.101	21.337
23	9.260	10.196	11.293	11.688	13.091	14.848	17.187	18.137	19.021	22.337
24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	19.037	19.943	23.337
25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	19.939	20.867	24.337
26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	20.843	21.792	25.336
27	11.808	12.879	14.125	14.573	16.151	18.114	20.703	21.749	22.719	26.336
28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	22.657	23.647	27.336
29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	23.567	24.577	28.336
30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	24.478	25.508	29.336

χ_α^2 – Critical Values of the Chi-squared Distribution

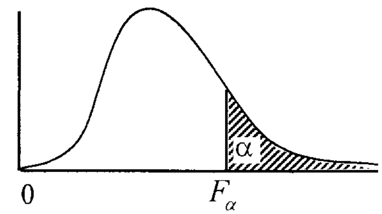
ν	α									
	0.30	**0.25**	**0.20**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.001**
1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.268
4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.465
5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.517
6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.322
8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.125
9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
11	12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
12	14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909
13	15.119	15.984	16.985	19.812	22.362	24.736	25.472	27.688	29.819	34.528
14	16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.123
15	17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.697
16	18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
17	19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.790
18	20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
19	21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.820
20	22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.315
21	23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.797
22	24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268
23	26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728
24	27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179
25	28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.620
26	29.246	30.434	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.052
27	30.319	31.528	32.912	36.741	40.113	43.194	44.140	46.963	49.645	55.476
28	31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.993	56.893
29	32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.336	58.302
30	33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.703

V_2	
	1
1	161.4
2	18.51
3	10.13
4	7.71
5	6.61
6	5.9
7	5.5
8	5.32
9	5.12
10	4.9
11	4.84
12	4.7
13	4.67
14	4.6
15	4.5
16	4.4
17	4.4
18	4.41
19	4.3
20	4.3
21	4.3
22	4.3
23	4.2
24	4.2
25	4.2
26	4.2
27	4.2
28	4.2
29	4.1
30	4.1
40	4.0
60	4.0
120	3.9
∞	3.8

5	0.001
79	10.827
97	13.815
38	16.268
60	18.465
50	20.517
48	22.457
78	24.322
55	26.125
89	27.877
88	29.588
57	31.264
00	32.909
19	34.528
19	36.123
01	37.697
57	39.252
18	40.790
56	42.312
32	43.820
97	45.315
01	46.797
96	48.268
31	49.728
58	51.179
28	52.620
90	54.052
45	55.476
93	56.893
36	58.302
72	59.703

6. Critical Values of the F-Distribution

Values of $F_{0.05}(v_1, v_2)$



ν_2	ν_1								
	1	2	3	4	5	6	7	8	9
1	161.4	199.5	21.57	224.6	230.2	234.0	236.8	238.9	240.5
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

Critical Values of the F-Distribution

Values of $F_{0.05}(\nu_1, \nu_2)$

ν_2	ν_1									
	10	12	15	20	24	30	40	60	120	∞
1	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.30	2.12	2.08	2.07	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.75	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Critical

ν_2	
	1
1	4052
2	98.50
3	34.12
4	21.20
5	16.26
6	13.75
7	12.25
8	11.26
9	10.56
10	10.04
11	9.65
12	9.33
13	9.07
14	8.86
15	8.68
16	8.53
17	8.40
18	8.29
19	8.18
20	8.10
21	8.02
22	7.95
23	7.88
24	7.82
25	7.77
26	7.72
27	7.68
28	7.64
29	7.60
30	7.56
40	7.31
60	7.08
120	6.85
∞	6.63

0	∞
.3	254.3
49	19.50
55	8.53
66	5.63
40	4.36
70	3.67
27	3.23
97	2.93
75	2.71
58	2.54
45	2.40
34	2.30
25	2.21
18	2.13
11	2.07
06	2.01
01	1.96
97	1.92
93	1.88
90	1.84
87	1.81
84	1.78
81	1.76
79	1.73
77	1.71
75	1.69
73	1.67
71	1.65
70	1.64
68	1.62
58	1.51
47	1.39
35	1.25
22	1.00

Critical Values of the F-Distribution

Values of $F_{0.01}\,(\nu_1, \nu_2)$

ν_2	ν_1								
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	4052	49995	5403	5625	5764	5859	59.28	5981	6022
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.46
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

Critical Values of the F-Distribution

Values of $F_{0.01}(\nu_1, \nu_2)$

ν_2	ν_1 10	12	15	20	24	30	40	60	120	∞
1	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.74	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

7. Fishe

r	0
0.0	0.
0.1	0.
0.2	0.
0.3	0.
0.4	0.
0.5	0.
0.6	0.
0.7	0.
0.8	1.
0.9	1.

* For negativ verse.

Confidence Level
$Z_{\alpha/2}$

8. Refer

α %
α
$-Z_{\alpha/2}$ an $+Z_{\alpha/2}$ for T.T.T
$-Z_\alpha$ for L.O.T.T
Z_α for R.O.T.T

∞
6366
99.50
26.13
13.46
9.02
6.88
5.65
4.86
4.31
3.91
3.60
3.36
3.17
3.00
2.87
2.75
2.65
2.57
2.49
2.42
2.36
2.31
2.26
2.21
2.17
2.13
2.10
2.06
2.03
2.01
1.80
1.60
1.38
1.00

7. Fisher's Z-Transformation

Values of $Z = \frac{1}{2} \ln \frac{1+r}{1-r}$

r	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.000	0.010	0.020	0.030	0.040	0.050	0.060	0.070	0.080	0.090
0.1	0.100	0.110	0.121	0.131	0.141	0.151	0.161	0.172	0.182	0.19
0.2	0.203	0.213	0.224	0.234	0.245	0.255	0.266	0.277	0.288	0.25
0.3	0.310	0.321	0.332	0.343	0.354	0.365	0.377	0.388	0.400	0.41
0.4	0.424	0.436	0.448	0.460	0.472	0.485	0.497	0.510	0.523	0.53
0.5	0.549	0.563	0.576	0.590	0.604	0.618	0.633	0.648	0.662	0.6
0.6	0.693	0.709	0.725	0.741	0.758	0.775	0.793	0.811	0.829	0.8
0.7	0.867	0.887	0.908	0.929	0.950	0.973	0.996	1.020	1.045	1.0
0.8	1.099	1.127	1.157	1.188	1.221	1.256	1.293	1.333	1.376	1.4
0.9	1.472	1.528	1.589	1.658	1.738	1.832	1.946	2.092	2.298	2.6

* For negative values of r put a minus sign in front of the corresponding Z's, and vice verse.

Confidence Level	99.73%	99%	98%	96%	95.50%	95%	90%	80%	68.27%	50%
$Z_{\alpha/2}$	3.00	2.58	2.33	2.05	2.00	1.96	1.645	1.28	1.00	0.675

8. Reference table of critical Values for a given L.O.S α

For T.T.T, R.O.T.T. And L.O.T.T

α % α	15% 0.15	10% 0.1	5% 0.05	4% 0.04	1% 0.01	0.5% .005	0.2% .002
$-Z_{\alpha/2}$ and $+Z_{\alpha/2}$ for T.T.T	-1.44 and 1.44	-1.645 and 1.645	-1.96 and 1.96	-2.06 and 2.06	-2.58 and 2.58	-2.81 and 2.81	-3.08 and 3.08
$-Z_\alpha$ for L.O.T.T	-1.04	-1.28	-1.645	-2.6	-2.33	-2.58	-2.88
Z_α for R.O.T.T	1.04	1.28	1.645	2.6	2.33	2.58	2.88

Index